1. (a) $(P_1 \wedge P_2) \to (P_1 \vee P_2)$ is a tautology:

$[\![P_1]\!]$	$[\![P_2]\!]$	$[\![(P_1$	$\wedge$	$P_2)$	$\to$	$(P_1$	$\vee$	$P_2)]\!]$
0	0	0	0	0	1	0	0	0
0	1	0	0	1	1	0	1	1
1	0	1	0	0	1	1	1	0
1	1	1	1	1	1	1	1	1

1 in all models.

(b) $(P_1 \vee P_2) \to (P_1 \wedge P_2)$ is <u>not</u> a tautology: e.g. in the model with $P_1^v = 1$, $P_2^v = 0$,

$$[\![(P_1 \vee P_2) \to (P_1 \wedge P_2)]\!]^v = 1 \to 0 = 0.$$

2. (a) $FV(\forall x_3 (P_1(x_2, x_3) \to x_3 \doteq x_2)) = \{x_2\}$

(b) $FV(\neg \exists x_1 \forall x_2\ f_3(x_2) \doteq x_1) = \emptyset.$

3. Take φ to be P_1, and ψ to be $\neg P_1$.

Then $\{P_1\}$ is consistent, with model $P_1^v = 1$,

$\{\neg P_1\}$ is consistent, - - - - - $P_1^v = 0$,

but $\{P_1, \neg P_1\}$ is inconsistent by soundness: $\dfrac{\neg P_1 \quad P_1}{\perp} \to E$

4. Correctly labelled derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{[\neg\varphi]^2 \qquad [\varphi]^1}{\bot}\ \cancel{\neg E}\ \to E}{\neg\neg\varphi}\ \cancel{RAA_2}\ \to I_2}{\varphi \to \neg\neg\varphi}\ \to I_1 \qquad \dfrac{\dfrac{\dfrac{[\neg\neg\varphi]^3 \qquad [\neg\varphi]^4}{\bot}\ \cancel{\neg E}\ \to E}{\varphi}\ RAA_4}{\neg\neg\varphi \to \varphi}\ \to I_3}{\varphi \leftrightarrow \neg\neg\varphi}\ \cancel{\leftrightarrow I}\ \wedge I$$

5. Recall from the course:

if $\varphi \vdash \psi$ and $\psi \vdash \varphi$,

then (by soundness) $\varphi \approx \psi$.

So we'll show $\forall x_1\,(P_1(x_1) \wedge P_2(x_1)) \vdash (\forall x_1\, P_1(x_1)) \wedge (\forall x_1\, P_2(x_1))$

and vice versa.

$$\dfrac{\dfrac{\dfrac{\dfrac{\forall x_1\,(P_1(x_1)\wedge P_2(x_1))}{P_1(x_1)\wedge P_2(x_1)}}{P_1(x_1)}\ \wedge E_L}{\forall x_1\, P_1(x_1)}\ \forall I \qquad \dfrac{\dfrac{\dfrac{\forall x_1\,(P_1(x_1)\wedge P_2(x_1))}{P_1(x_1)\wedge P_2(x_1)}\ \forall E}{P_2(x_1)}\ \wedge E_R}{\forall x_1\, P_2(x_1)}\ \forall I}{(\forall x_1\, P_1(x_1)) \wedge (\forall x_1\, P_2(x_1))}\ \wedge I$$

5 (cont'd)

$$\dfrac{\dfrac{\dfrac{\dfrac{(\forall x_1\, P_1(x_1))\land(\forall x_1\, P_2(x_1))}{\forall x_1\, (P_1(x_1))}\land E_L}{P_1(x_1)}\forall E \qquad \dfrac{\dfrac{(\forall x_1\, P_1(x_1))\land(\forall x_1\, P_2(x_1))}{\forall x_1\, P_2(x_1)}\land E_R}{P_2(x_1)}\forall E}{P_1(x_1)\land P_2(x_1)}\land I}{\forall x_1\, (P_1(x_1)\land P_2(x_1))}\forall I$$

6. (a) This <u>fails</u>.

Countermodel: any non-injective function,

e.g. $|A| = \mathbb{Z}$, $f_1^A(n)=0$ for all n (so f_1 is the <u>constant</u> 0 function),

$v(x_1) = 0, \quad v(x_2) = 1$

gives $[\![\neg(x_1 \doteq x_2)]\!]^{A,v} = \cancel{0}\ 1$,

$[\![\neg(f_1(x_1) \doteq f_1(x_2))]\!]^{A,v} = 0$,

so $\neg(x_1 \doteq x_2) \not\models \neg(f_1(x_1) \doteq f_1(x_2))$,

so by soundness, --- $\nvdash$ ------.

(b) This <u>holds</u>. Derivation:

$$\dfrac{\dfrac{\neg(f_1(x_1)\doteq f_1(x_2)) \qquad \dfrac{[x_1\doteq x_2]^1 \qquad \dfrac{}{f_1(x_1)\doteq f_1(x_1)}\text{refl}}{f_1(x_1)\doteq f_1(x_2)}\text{repl}}{\bot}\to E}{\neg(x_1\doteq x_2)}\to I_1$$

(with ~~formula~~ substitution $(f_1(x_1)\doteq f_1(x_3))$ $[x_1/x_3]$, & $[x_2/x_3]$.)

7. In soundness, we are proving by induction on derivations $\mathcal{D}$ the statement "for any interpretation $\mathcal{A}, v$, if all undisch. assumptions of $\mathcal{D}$ hold in $\mathcal{A}, v$, then the conclusion of $\mathcal{D}$ holds in $\mathcal{A}, v$." } "soundness for $\mathcal{D}$"

(a) Case: $\mathcal{D}$ is of the form

$$\mathcal{D}\left\{\begin{array}{c} \mathcal{D}'\left\{\begin{array}{c}[\varphi]\\ \vdots \\ \psi\end{array}\right. \\ \hline \varphi \to \psi \end{array} \to I\right.$$

Inductive hypothesis (IH): soundness holds for $\mathcal{D}'$.
Let $\mathcal{A}, v$ be any interpretation in which all undisch. ass'ns of $\mathcal{D}$ hold; we need to show $\mathcal{A}, v \models \varphi \to \psi$.

The undisch. ass'ns of $\mathcal{D}'$ are those of $\mathcal{D}$, plus possibly also φ. So if φ holds in $\mathcal{A}, v$, then all undisch. ass'ns of $\mathcal{D}'$ hold, so by soundness for $\mathcal{D}'$ (the IH), ψ holds in $\mathcal{A}, v$, so $[\![\varphi \to \psi]\!]^{\mathcal{A},v} = 1 \to 1 = 1$.
On the other hand, if φ doesn't hold in $\mathcal{A}, v$, then $[\![\varphi \to \psi]\!]^{\mathcal{A},v} = 0 \to [\![\psi]\!]^{\mathcal{A},v} = 1$.

So in either case, $\varphi \to \psi$ holds in $\mathcal{A}, v$, as required.

7 (b) <u>Case:</u> $\mathcal{D}$ is of the form

$$\mathcal{D}\left\{\begin{array}{c} \mathcal{D}'\left\{\begin{array}{c}\vdots \\ \forall x_i\, \varphi\end{array}\right. \\ \hline \varphi[t/x_i]\end{array}\;\forall E\right.$$

with t free for x_i in φ.

IH: Soundness holds for $\mathcal{D}'$.

Take any interp'n $\mathcal{A}, v$ where all undischd assns of $\mathcal{D}$ hold. $\forall E$ discharges no assumptions, so all assns of $\mathcal{D}'$ are assns of $\mathcal{D}$, so hold in $\mathcal{A}, v$.

So we can apply soundness for $\mathcal{D}'$ (the IH) to see that $\mathcal{A}, v \models \forall x_i\, \varphi$,

i.e. for all $a \in A$, $[\![\varphi]\!]^{\mathcal{A}, v[x_i \mapsto a]} = 1$.

In particular, taking $a = [\![t]\!]^{\mathcal{A},v}$ gives

$$[\![\varphi[t/x_i]]\!]^{\mathcal{A},v} = [\![\varphi]\!]^{\mathcal{A}, v[x_i \mapsto [\![t]\!]^{\mathcal{A},v}]} \quad \text{(by the given lemma)}$$

$$= 1 \quad \text{as required.}$$

8. Derivation of $\underbrace{(\forall x_1 \exists x_2\ \neg(x_1 \doteq x_2))}_{\varphi_1} \longleftrightarrow \underbrace{(\exists x_3, x_4\ \neg(x_3 \doteq x_4))}_{\varphi_2}$

$$\dfrac{\begin{matrix} ①\ \vdots \\ \varphi_1 \to \varphi_2 \end{matrix} \qquad \begin{matrix} ②\ \vdots \\ \varphi_2 \to \varphi_1 \end{matrix}}{\varphi_1 \leftrightarrow \varphi_2}\ \wedge I$$

①:

$$\dfrac{\dfrac{\dfrac{\dfrac{[\forall x_1 \exists x_2\ \neg(x_1 \doteq x_2)]^1}{\exists x_2\ \neg(x_3 \doteq x_2)}\ \forall E \text{ (with term } x_3) \qquad \dfrac{[\neg x_3 = x_2]^2}{\exists x_4\ \neg(x_3 \doteq x_4)}\ \exists I \text{ (with term } x_2)}{\exists x_4\ \neg(x_3 \doteq x_4)}\ \exists E_2}{\exists x_3, x_4\ \neg(x_3 \doteq x_4)}\ \exists I \text{ (with term } x_3)}{\varphi_1 \to \varphi_2}\ \to I_1$$

②: see next page

8 (a) cont'd.

(with term x_4)

$$[\neg(x_1 \doteq x_4)]^5 \quad \exists I$$

just like derivation of '$x_1 = x_4$' to right

$$\neg\exists x_2 \neg(x_1 \doteq x_2) \qquad \exists x_2 \neg(x_1 \doteq x_2) \quad \to E$$

$$\perp$$

$$x_1 = x_3 \;\; \mathrm{RAA}_4 \qquad x_1 = x_4 \;\; \mathrm{RAA}_5$$

$$[\neg x_3 \doteq x_4]^3 \qquad x_3 \doteq x_4 \quad \text{repl.}$$

$$\perp$$

$$[\exists x_4 \neg(x_3 \doteq x_4)]^2 \qquad \exists x_2 \neg(x_1 \doteq x_2) \quad \mathrm{RAA}$$

$$[\exists x_3, x_4 \neg(x_3 \doteq x_4)]^1 \qquad \exists x_2 \neg(x_1 \doteq x_2) \quad \exists E_3$$

$$\exists x_2 \neg(x_1 \doteq x_2) \quad \exists E_2$$

$$\forall x_1 \exists x_2 \neg(x_1 \doteq x_2) \quad \forall I$$

$$\varphi_2 \to \varphi_1 \quad \to I_1$$

(b) By completeness, to show $\vdash \varphi_1 \leftrightarrow \varphi_2$ it suffices to show $\vDash \varphi_1 \leftrightarrow \varphi_2$; equivalently, that in every ~~structure~~ interpretation $\mathcal{A}, v$, $[\![\varphi_1]\!]^{\mathcal{A},v} = [\![\varphi_2]\!]^{\mathcal{A},v}$.

So: take some structure $\mathcal{A}$, & valuation v.

If $|\mathcal{A}|$ has ≥ 2 elements, then certainly

$$[\![\varphi_1]\!] = [\![\varphi_2]\!] = 1.$$

If $|\mathcal{A}|$ has exactly 1 element, then similarly $[\![\varphi_1]\!] = [\![\varphi_2]\!] = 0$.

And $|\mathcal{A}|$ can't be empty, since it contains e.g. $v(x_0)$.

So in any case, we have $[\![\varphi_1]\!] = [\![\varphi_2]\!]$ as required.

9 (a) The theory $\Gamma_1 = \{\forall x_1\ x_1 \doteq f_1\}$, over the arity type $\langle\ ;0\rangle$ (i.e. a single constant symbol f_1) is modelled by any <u>singleton</u> set, but has no other models — in particular, no infinite model.

(b) Work over the arity type $\langle\ ;0,1\rangle$: a constant symbol f_1 & a unary function f_2.

Take the theory $\Gamma_2 = \{ \overbrace{\forall x_1, x_2\ (f_2(x_1) \doteq f_2(x_2) \to x_1 \doteq x_2)}^{\varphi_{inj}},$
$\underbrace{\neg \exists x_1\ f_2(x_1) \doteq f_1}_{\varphi_{nonsurj}} \}$.

Then in any model A of Γ_2,
f_2^A must be <u>injective</u>, since $A \models \varphi_{inj}$,
& f_1^A <u>can't be in the image</u> of f_2^A, since $A \models \varphi_{nonsurj}$,
so $f_1^A,\ f_2^A(f_1^A),\ f_2^A(f_2^A(f_1^A))\ f_2^A(f_2^A(f_2^A(f_1^A))), \ldots$
gives an infinite sequence of distinct elements of A.
So every such model A is infinite.

(Alternate answer: take $\Gamma_2 = \{\sigma_1, \sigma_2, \sigma_3, \ldots\}$
$= \{\sigma_n \mid n \in \mathbb{N}\}$,
where σ_n asserts existence of $\geq n$ elts, as given in the question.)

9 (b). Suppose Γ is a theory with arbitrarily large finite models.

Take Γ' to be $\Gamma \cup \{\sigma_n \mid n \in \mathbb{N}\}$,

(where σ_n asserts existence of $\geq n$ elts, as given in the question).

Any finite subset $\Delta \subseteq \Gamma'$ is contained within some theory of the form $\Gamma \cup \{\sigma_n \mid 0 \leq n \leq m\}$, for some $m \in \mathbb{N}$ thus any model of Γ with at least m elements is a model of Δ, so Δ has some model, so by soundness, Δ is consistent.

So by compactness Γ' is consistent; so by completeness it has some model. But such a model must be an infinite model of Γ.

10 (a) $\exists x_2 \forall x_1 \varphi \vdash \forall x_1 \exists x_2 \varphi$:

$$\dfrac{\exists x_2 \forall x_1 \varphi \qquad \dfrac{\dfrac{[\forall x_1 \varphi]^1}{\varphi}\,\forall E}{\exists x_2 \varphi}\,\exists I}{\dfrac{\exists x_2 \varphi}{\forall x_1 \exists x_2 \varphi}\,\forall I}\,\exists E_1$$

(b) Take φ to be $P_1(x_1, x_2)$ (in similarity type with a single binary relation symbol P_1).

Then the structure $\mathcal{A}$ with $|\mathcal{A}| = \mathbb{N}$,

$P_1^{\mathcal{A}} = \{(x_1, x_2) \mid x_1 < x_2\}$

shows that $\forall x_1 \exists x_2 P_1(x_1, x_2) \not\models \exists x_2 \forall x_1 P_1(x_1, x_2)$

(since for all $n_1 \in \mathbb{N}$, there's some $n_2 \in \mathbb{N}$ s.t. $n_1 < n_2$, but there's no n_2 s.t. for all n_1, $n_1 < n_2$)

so by soundness, $\forall x_1 \exists x_2 P_1(x_1, x_2) \nvdash \exists x_2 \forall x_1 P_1(x_1, x_2)$.

(c) Take φ to be $P_1(x_1)$ (or any other non-tautology in which x_2 is not free).

Then we can derive $\forall x_1 \exists x_2 \varphi \vdash \exists x_2 \forall x_1 \varphi$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\forall x_1 \exists x_2\, \varphi}{\exists x_2\, \varphi}\ \cancel{\forall E} \qquad [\varphi]^1}{\varphi}\ \exists E_1\ \circledast}{\forall x_1\, \varphi}\ \forall I}{\exists x_2 \forall x_1\, \varphi}\ \exists I$$

$\circledast$ Note this use of $\exists E$ is valid since x_2 not free in φ.

On the other hand, $\exists x_2 \forall x_1 P_1(x_1)$ is not a tautology: ~~[illegible]~~ any model such that $P_1^{\mathcal{A}}$ is empty gives a countermodel.