Logic Exam Solution 2022-08-11

1) This is a theory question refer to the textbook

2)

a) $\neg P_2$

b) $P_1 \vee \neg P_3$

This is true in v_1 because $P_1^{v_1} = 1$

This is true in v_2 and v_3 because $P_3^{v_2} = P_3^{v_3} = 0$

This is not a tautology because it is false when $P_1^{v} = 0$ and $P_3^{v} = 1$

3) a) A countermodel is given by v with

$P_3^{v} = 1$ $\quad$ $P_2^{v} = 0$

In this model $[P_1 \to P_2]^{v} = 1$

but $[P_2 \to P_1]^{v} = 0$

b)

$$\dfrac{\dfrac{\dfrac{\dfrac{P_1 \to P_2 \qquad [P_1]^1}{P_2}\,{\to}E \qquad [\neg P_2]}{\bot}}{\neg P_1}\,{\to}I_1}{\neg P_2 \to \neg P_1}\,{\to}I_2$$

4) (a)

$FV([P_1(x_1, x_{23}) \to P_2(x_1)] \vee [P_2(x_1) \to \forall x_{47} P_1(x_{47}, x_5)])$

$= FV(P_1(x_1, x_{23}) \to P_2(x_1)) \cup FV(P_2(x_1) \to \forall x_{47} P_1(x_{47}, x_5))$

$= \{x_1, x_{23}\} \cup \{x_1\} \cup (\{x_{47}, x_5\} \setminus \{x_{47}\}) =$

$= \{x_1, x_{23}, x_5\}$

b) $FV(\forall x_1 (P_3(x_1) \to \exists x_2 (P_5(x_2) \wedge P_{17}(x_2, x_1, x_2, x_5)))) =$

$= FV(P_5(x_1) \to \exists x_2 (P_3(x_2) \wedge P_{17}(x_2, x_1, x_2, x_5))) \setminus \{x_1\}$

$= (\{x_1\} \cup FV(\exists x_2 (P_3(x_2) \wedge P_{17}(x_2, x_1, x_2, x_5)))) \setminus \{x_1\}$

$= (\{x_1\} \cup (\{x_2, x_1, x_5\} \setminus \{x_2\})) \setminus \{x_1\}$

$= \{x_5\}$

5) (a) This derivation is correct (x_0 is free for x_1 in P so we can substitute)

(b) This derivation is not correct. The problem is in the use of $\forall I$: in order to use that x_1 should not occur freely in any of the OA

6) (a) To show consistency it is enough to provide a model

$P_0^v = 1$ $P_1^v = 0$ works so it is consistent

(b) It is finite so it cannot be maximally consistent

7) (a) The arity is (2 1) (1 binary relation and one binary operation

(b) The "translation" of the formula in this interpretation is

$\forall x_1 x_2 x_3 \quad x_1 > x_2 \rightarrow x_1 + x_3 > x_2$

which is clearly false

$1 > 0$ is true but

$1 + (-2) > 0$ is false.

$[P_1(x_1 x_2) \rightarrow P(f_1(x_1 x_3) x_2)]^{V(x_1 \to 0)(x_2 \to 1)(x_3 \to -2)} = 0$

8) φ_{ref} correspond to the property P_1 is reflexive

φ_{trans} correspond to the property P_1 is transitive

φ_{ord} correspond to be a partial ore

φ_{tot} says that given any two elements they are in relation up to the ord

All 4 are true iff P_1 is a total orde.

(a) Thus φ_{os} $\langle \mathbb{N} \ \leq \rangle$ is a model as this is a totally ordered set

(b) This is not true think about $\equiv$ (mod 3) this is a reflexive & transitive relation

but $0 \equiv 3 \pmod 3$

$3 \equiv 0 \pmod 3$ but $0 \neq 3$

$(\mathbb{N}; ; \equiv_3)$ is a countermodel

(c) This is false $X = \{a, b\}$

$(\mathcal{P}(X); ; \subseteq)$

↳ this is a order relation so φ_{refl} φ_{tran} φ_{aso} are true

but the order is not total

$\{a\} \not\subseteq \{b\} \wedge \{b\} \not\subseteq \{a\}$

$\Longrightarrow$ φ_{tot} is false

9) (a) SEMANTIC ARGUMEN Let $\mathcal{A}$ be a system such that $\mathcal{A} \models \exists x_2 \forall x_1 \varphi$

there is an $a \in A$ such that

$$[\![\forall x_1 \varphi]\!]^{\mathcal{A}\,\sigma[x_2 \to a]} = 1$$

So, for every $b \in A$ $[\![\varphi]\!]^{\mathcal{A}\,\sigma[x_2 \to a][x_1 \to b]} = 1$

$$\| \quad [\![\varphi]\!]^{\mathcal{A}\,\sigma[x_1 \to b][x_2 \to a]}$$

for $a \in A$ and for every $b \in A$

$$[\![\exists x_2 \varphi]\!]^{\mathcal{A}\sigma[x_1 \to b]} = 1$$ for every $b \in A$

therefore

$$[\![\forall x_1 \exists x_2 \varphi]\!]^{\mathcal{A}\sigma} = 1$$

DERIVATION

$$\dfrac{\dfrac{\exists x_2 \forall x_1 \varphi \qquad \dfrac{\dfrac{[\forall x_1 \varphi]^1}{\varphi}\,\forall E}{\exists x_2 \varphi}\,\exists I}{\exists x_2 \varphi}\,\exists E^1}{\forall x_1 \exists x_2 \varphi}\,\forall I$$

b) $\varphi = \text{"}f_1(x_1\ x_2) \doteq f_0\text{"}$

$\langle \mathbb{Z}; +; 0 \rangle$ $\quad \forall x_1 \exists x_2 \varphi$ is true

(every element has it opposite)

$\exists x_1 \forall x_2 \varphi$ there is $a \in \mathbb{Z}$ such that for every $b \in \mathbb{Z}$

$a + b = 0$

c) If $\varphi = P_1(x_1)$ we have the following derivatio

$$\dfrac{\dfrac{\dfrac{\dfrac{\forall x_1 \exists x_2 P_1(x_1)}{\exists x_2 P_1(x_1)}\,\forall E \qquad [P_1(x_1)]^1}{P_1(x_1)}\,\exists E}{\forall x_1 P_1(x_1)}\,\forall I}{\exists x_2 \forall x_1 P_1(x_1)}\,\exists I$$

10) (a) We prove the contrapositive

Suppose $\Gamma \nvdash \varphi$ then

$\Gamma \cup \{\neg\varphi\}$ is consistent

otherwise $\Gamma, \neg\varphi \vdash \bot$

$$\dfrac{\dfrac{\begin{matrix}\Gamma \quad [\neg\varphi] \\ \vdots\end{matrix}}{\bot}}{\varphi}\ \text{RAA} \qquad \Gamma \vdash \varphi$$

by the Model existing Lemma we have that

$\Gamma \cup \{\neg\varphi\}$ has a model

but then $\Gamma \not\models \varphi$

(b) PROPOSITIONAL LOGIC

Let Γ consistent can assume max cons

define $V_\Gamma(P_i) = \begin{cases} 1 & \text{is } \Gamma \vdash P_i \\ 0 & \text{otherwise} \end{cases}$

By induction as in the lectures

$V_\Gamma \models \varphi \iff \Gamma \vdash \varphi$

If $\varphi \in \Gamma$ then $\Gamma \models \varphi$ by completeness

$\Gamma \vdash \varphi$ so $V_\Gamma(\varphi) = 1$

PREDICATE LOGIC

Again we can assume Γ is max consistent

As in the lecture defin

$t \sim s \iff t \doteq s \in \Gamma$

this is an equivalence relation

$\mathcal{A}_\Gamma = \langle Term/_\sim, P_i^{\mathcal{A}}, f_i^{\mathcal{A}} \rangle$ As in the lectu

$v^* : Var \longrightarrow Term/_\sim$

$x_i \longmapsto [x_i]$

We have that $\mathcal{A}_\Gamma v^* \models \varphi \iff \Gamma \vdash \varphi$

$\varphi \in \Gamma \Rightarrow \Gamma \models \varphi \overset{c}{\Rightarrow} \Gamma \vdash \varphi$

So $\mathcal{A}_\Gamma v^* \models \varphi$

This is a model for Γ

ALTERNATIVE

suppose Γ has no model. Then

$\Gamma \models \bot$

By completeness $\Gamma \vdash \bot$

So Γ is inconsistent