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Matematiska institutionen
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Written Exam/Skriftligt prov
Logic 7.5 hp / Logik 7,5 hp
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No aids are allowed except for paper, pen and the list of formulas for inductive proofs. Write clearly and motivate your answers carefully. It is preferred that you write your answers in English, but Swedish is also allowed.

This exam consists of two parts. The Basic part has 8 problems (1–8) worth a total of 20 points. The Problem part has 4 problems (9–13) worth a total of 20 points. You can obtain a maximum of 40 points and if you score at least 20 points in total it is a passing grade.

Written Exam

Basic part

1. Let C be a subset of the natural numbers $\mathbb{N}$ inductively defined by

- $3 \in C$;
- if $n \in C$ then $3n \in C$.

(a) Explain in words what the set C consists of, that is to say, give a *non-inductive* definition of the set C.

(b) Give a proof using induction of your answer in (a). That is, let D be the set that you have defined directly in part (a), which you claim is equal to C. Prove by induction that $C \subseteq D$. Be rigorous. Then give at least a short informal argument that $D \subseteq C$.

(c) Give a non-recursive definition of the function $f : C \to \mathbb{N}$ defined recursively by

$$\begin{aligned} f(3) &= 1 \\ f(3n) &= f(n) + 1. \end{aligned}$$

3 p

2. Let P_1 and P_2 be propositional variables, and let $\mathcal{A}$ be an interpretation such that $P_1^{\mathcal{A}}$ and $P_2^{\mathcal{A}}$ are both true.

(a) Compute the truth-value of $((\neg P_1) \vee (\neg P_2)) \to \neg(P_1 \wedge P_2)$; 2 p

(b) Based on your answer in (a), can you say whether the formula in (a) is valid or not? Why/why not? 1 p

3. Give a natural deduction proof of $((\neg\phi) \vee (\neg\psi)) \to \neg(\phi \wedge \psi)$.

2 p

4. Let Γ be a set of propositional formulas and let ϕ be a propositional formula.

(a) What does $\Gamma \vdash \phi$ mean? What does $\Gamma \vDash \phi$ mean? State both definitions very carefully. 2 p

(b) State the soundness theorem and the completeness theorem for propositional logic. 2 p

5. In the language with only equality $\mathcal{L} = (\langle ; \rangle)$ consider the sentence

$$\forall x \forall y \forall z (z \doteq x \vee z \doteq y).$$

Give an interpretation $\mathcal{A}$ where this sentence is true. Give another interpretation $\mathcal{B}$ where the sentence is false.

2 p

6. Consider the language with only one unary function symbol $(\langle ; f \rangle)$.

(a) Write a sentence in $\mathcal{L}$ stating that f is injective (1-1).

(b) Write a sentence in $\mathcal{L}$ stating that f is surjective (onto).

2 p

7. In the formula $\exists x_0 \forall x_1 \exists x_2 \forall x_5 (R_1(x_0, x_1, x_3) \vee (f_1(x_3, x_2) \doteq f_2(x_2, x_3))$ which variables are free and which are bound? Is the term $f_0(x_0, x_7, x_3)$ free for x_3 in this formula? 2 p

8. What does it mean that a sentence in predicate logic is a *tautology*? Prove carefully that the sentence that you wrote in part 6 (a) is not a tautology. 2 p

Problem part

9. Provide derivations in natural deduction without any undischarged assumptions of the following formulas:

(a) $(\phi \wedge (\psi \vee \sigma)) \rightarrow ((\phi \wedge \psi) \vee (\phi \wedge \sigma))$ 2 p

(b) $\exists x (\phi \vee \psi) \rightarrow (\exists x \phi \vee \exists x \psi)$ 3 p

10. For each of the following formulas decide, using a method of your choice, whether it is derivable in natural deduction (for general formulas ϕ, ψ) and justify your answer:

(a) $(\exists x \phi \wedge \exists x \psi) \rightarrow \exists x (\phi \wedge \psi)$ 2 p

(b) $(\forall x \phi \vee \forall x \psi) \rightarrow \forall x (\phi \vee \psi)$ 2 p

11. Let Γ be a set of sentences and ϕ a sentence. Show that if $\Gamma \cup \{\phi\}$ is inconsistent, then $\Gamma \vdash \neg \phi$. 2 p

12. Do the following:

(a) State the compactness theorem for predicate logic. 1 p

(b) Let $\mathcal{L} = (\langle ; \rangle)$ be the language with only equality. Prove that there does not exist a sentence ϕ in $\mathcal{L}$ such that ϕ is true in a structure if and only if that structure is finite. (You may take as given that there exists, for any number n, a sentence ψ_n saying that there are at least n elements). 3 p

(c) Conclude that there does not exist a sentence ϕ in $\mathcal{L}$ such that ϕ is true in a structure if and only if that structure is infinite. 1 p

13. In the language with only equality, it is the case that if a sentence is true in one infinite structure, then it is true in all infinite structures (you are not supposed to prove this!). Thus from 12 (c) we can conclude that there exists no sentence in the language of equality such that 1) there exists a structure in which the sentence is true, and 2) if the sentence is true in a structure then that structure must be infinite.

Show that if the language contains a unary function symbol, then this is no longer true. That is, show that in the language $\mathcal{L} = (\langle ; f \rangle)$ with only one unary function symbol, there exists a sentence ϕ such that 1) ϕ is true in some structure, and 2) for every structure, if ϕ is true in it then that structure must be infinite. (Hint: Consider the sentence you wrote in 6 (a) and the negation of the sentence you wrote in 6 (b)). 4 p

——— Good luck ———