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Solutions
Logic 7.5 hp / Logik 7,5 hp
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Basic part

1. Let C be a subset of the natural numbers $\mathbb{N}$ inductively defined by

- $3 \in C$;
- if $n \in C$ then $3n \in C$.

(a) Explain in words what the set C consists of, that is to say, give a *non-inductive* definition of the set C.

(b) Give a proof using induction of your answer in (a). That is, let D be the set that you have defined directly in part (a), which you claim is equal to C. Prove by induction that $C \subseteq D$. Be rigorous. Then give at least a short informal argument that $D \subseteq C$.

(c) Give a non-recursive definition of the function $f : C \to \mathbb{N}$ defined recursively by

$$\begin{aligned} f(3) &= 1 \\ f(3n) &= f(n) + 1. \end{aligned}$$

3 p

Solution.

(a) The set C consists of all positive powers 3, i.e, numbers of the form 3^k for $k \in \mathbb{N} \setminus \{0\}$.

(b) We prove by induction on $n \in C$ that $n \in D = \{3^k \mid k = 1, 2, \ldots\}$. Clearly, $3 \in D$, so the base case holds. For the induction step, we assume that $n \in D$ and show that $3n \in D$. Since $n \in D$, we can find a positive integer k such that $n = 3^k$, in which case $3n = 3^{k+1}$ also belongs to D. This concludes the induction and shows that $C \subseteq D$. To show the converse inclusion, one needs to show that $3^k \in C$ for each $k = 1, 2, \ldots$, which is a simple proof by induction on k.

(c) If we denote the k-th element of C as $c_k = 3^k$ then $f(c_k) = \log_3 c_k = k$.

2. Let P_1 and P_2 be propositional variables, and let $\mathcal{A}$ be an interpretation such that $P_1^{\mathcal{A}}$ and $P_2^{\mathcal{A}}$ are both true.

(a) Compute the truth-value of $((\neg P_1) \vee (\neg P_2)) \to \neg(P_1 \wedge P_2)$; 2 p

(b) Based on your answer in (a), can you say whether the formula in (a) is valid or not? Why/why not? 1 p

Solution.

(a) If the values of the propositional variable P_1 and P_1 are both true in the interpretation $\mathcal{A}$ then $P_1 \wedge P_2$ is true and $\neg(P_1 \wedge P_2)$ is false. On the other hand, $\neg P_1$ and $\neg P_2$ are both false, and $\neg P_1 \vee \neg P_2$ is false. Thus $((\neg P_1) \vee (\neg P_2)) \to \neg(P_1 \wedge P_2)$ is true in $\mathcal{A}$.

(b) The answer to (a) is not enough to conclude if the formula is valid since one needs to check all values of $P_1^{\mathcal{A}}$ and $P_2^{\mathcal{A}}$ to conclude/reject this claim. However it is easy to check that the formula is a tautology, i.e. is true for all values of variables.

3. Give a natural deduction proof of $((\neg\varphi) \vee (\neg\psi)) \to \neg(\varphi \wedge \psi)$.

2 p

Solution.

$$\dfrac{\dfrac{\dfrac{[(\neg\varphi) \vee (\neg\psi)]^1 \qquad \dfrac{[\neg\varphi]^3 \quad \dfrac{[\varphi \wedge \psi]^2}{\varphi}\,\wedge E}{\bot}\,\to E \qquad \dfrac{[\neg\psi]^4 \quad \dfrac{[\varphi \wedge \psi]^2}{\psi}\,\wedge E}{\bot}\,\to E}{\bot}\,\vee E_{3,4}}{\neg(\varphi \wedge \psi)}\,\to I_2}{((\neg\varphi) \vee (\neg\psi)) \to \neg(\varphi \wedge \psi)}\,\to I_1$$

4. Let Γ be a set of propositional formulas and let φ be a propositional formula.

(a) What does $\Gamma \vdash \varphi$ mean? What does $\Gamma \models \varphi$ mean? State both definitions very carefully.

2 p

(b) State the soundness theorem and the completeness theorem for propositional logic.

2 p

Solution.

(a) The notation $\Gamma \vdash \varphi$ means that the theory Γ *proves* the formula φ, i.e. that there is a natural deduction ending in the formula φ whose undischarged assumptions are all formulas in Γ.
The notation $\Gamma \models \varphi$ means that Γ *models* φ, i.e. in any interpretation in which all formulas in Γ are true the formula φ is true as well.

(b) The soundness theorem in propositional logic means that for any (propositional) theory Γ and formula φ such that $\Gamma \vdash \varphi$, one has $\Gamma \models \varphi$.
The completeness theorem is the converse to the soundness theorem, i.e. it claims that in propositional logic $\Gamma \models \varphi$ implies $\Gamma \vdash \varphi$.

5. In the language with only equality $\mathcal{L} = (\langle ; \rangle)$ consider the sentence

$$\forall x \forall y \forall z (z = x \vee z = y).$$

Give an interpretation $\mathcal{A}$ where this sentence is true. Give another interpretation $\mathcal{B}$ where the sentence is false.

2 p

Solution.
This sentence is true in a structure precisely if the domain of that structure contains at most two elements. Hence, if $|\mathcal{A}| = \{0,1\}$ then the sentence is true in $\mathcal{A}$ whereas if $|\mathcal{B}| = \{0,1,2\}$ then the sentence is false in $\mathcal{B}$.

6. Consider the language with only one unary function symbol ($\langle ; f\rangle$).

(a) Write a sentence in $\mathcal{L}$ stating that f is injective (1-1).

(b) Write a sentence in $\mathcal{L}$ stating that f is surjective (onto).

2 p

Solution.

(a) $\forall x \forall y\ (f(x) \doteq f(y) \to x \doteq y)$

(b) $\forall x \exists y\ (x \doteq f(y))$

7. In the formula $\exists x_0 \forall x_1 \exists x_2 \forall x_5 (R_1(x_0, x_1, x_3) \vee (f_1(x_3, x_2) = f_2(x_2, x_3)))$ which variables are free and which are bound? Is the term $f_0(x_0, x_7, x_3)$ free for x_3 in this formula?

2 p

Solution.
The only free variable is x_3 whereas the bound variables are x_0, x_1, x_2, x_5. The term $f_0(x_0, x_7, x_3)$ is not free for x_3 in the formula since the variable x_0 is captured by $\exists x_0$.

8. What does it mean that a sentence in predicate logic is a *tautology*? Prove carefully that the sentence that you wrote in part 6 (a) is not a tautology.

2 p

Solution. A sentence is said to be a tautology if it is true in all interpretations. The formula in 6(a) is not a tautology since if one considers a structure $\mathcal{A}$ such that the function $f^{\mathcal{A}}$ is not injective then it will be false in that interpretation. For instance, one can take $|\mathcal{A}| = \mathbb{R}$ the function $x \mapsto x^2$ for $f^{\mathcal{A}}$.

Here is a careful justification. For $\forall x \forall y\ (f(x) \doteq f(y) \to x \doteq y)$ to be true, one needs

$$[\![f(x) \doteq f(y) \to x \doteq y]\!]^{[x \mapsto r_1][y \mapsto r_2]} = 1$$

for all reals r_1, r_2. But if one takes $r_1 = -1$ and $r_2 = 1$, then $[\![f(x) \doteq f(y)]\!]^{[x \mapsto -1][y \mapsto 1]} = 1$ but $[\![x \doteq y]\!]^{[x \mapsto -1][y \mapsto 1]} = 0$. This means that the whole sentence does not hold in this structure.

Problem part

9. Provide derivations in natural deduction without any undischarged assumptions of the following formulas:

(a) $(\varphi \wedge (\psi \vee \sigma)) \to ((\varphi \wedge \psi) \vee (\varphi \wedge \sigma))$ 2 p

(b) $\exists x(\varphi \vee \psi) \to (\exists x \varphi \vee \exists x \psi)$ 3 p

Solution.

(a)

$$
\begin{prooftree}
\AxiomC{$[\varphi \wedge (\psi \vee \sigma)]^1$}
\RightLabel{$\wedge E$}
\UnaryInfC{$\psi \vee \sigma$}
\AxiomC{$[\varphi \wedge (\psi \vee \sigma)]^1$}
\RightLabel{$\wedge E$}
\UnaryInfC{φ}
\AxiomC{$[\psi]^2$}
\RightLabel{$\wedge I$}
\BinaryInfC{$\varphi \wedge \psi$}
\RightLabel{$\vee I$}
\UnaryInfC{$(\varphi \wedge \psi) \vee (\varphi \wedge \sigma)$}
\AxiomC{$[\varphi \wedge (\psi \vee \sigma)]^1$}
\RightLabel{$\wedge E$}
\UnaryInfC{φ}
\AxiomC{$[\sigma]^3$}
\RightLabel{$\wedge I$}
\BinaryInfC{$\varphi \wedge \sigma$}
\RightLabel{$\vee I$}
\UnaryInfC{$(\varphi \wedge \psi) \vee (\varphi \wedge \sigma)$}
\RightLabel{$\vee E_{2,3}$}
\TrinaryInfC{$(\varphi \wedge \psi) \vee (\varphi \wedge \sigma)$}
\RightLabel{$\to I_1$}
\UnaryInfC{$(\varphi \wedge (\psi \vee \sigma)) \to ((\varphi \wedge \psi) \vee (\varphi \wedge \sigma))$}
\end{prooftree}
$$

(b)

$$
\begin{prooftree}
\AxiomC{$[\exists x(\varphi \vee \psi)]^1$}
\AxiomC{$[\varphi \vee \psi]^2$}
\AxiomC{$[\varphi]^3$}
\RightLabel{$\exists I$}
\UnaryInfC{$\exists x\varphi$}
\RightLabel{$\vee I$}
\UnaryInfC{$\exists x\varphi \vee \exists x\psi$}
\AxiomC{$[\psi]^4$}
\RightLabel{$\exists I$}
\UnaryInfC{$\exists x\psi$}
\RightLabel{$\vee I$}
\UnaryInfC{$\exists x\varphi \vee \exists x\psi$}
\RightLabel{$\vee E_{3,4}$}
\TrinaryInfC{$\exists x\varphi \vee \exists x\psi$}
\RightLabel{$\exists E_2$}
\BinaryInfC{$\exists x\varphi \vee \exists x\psi$}
\RightLabel{$\to I_1$}
\UnaryInfC{$\exists x(\varphi \vee \psi) \to (\exists x\varphi \vee \exists x\psi)$}
\end{prooftree}
$$

10. For each of the following formulas decide, using a method of your choice, whether it is derivable in natural deduction (for general formulas φ, ψ) and justify your answer:

(a) $(\exists x\varphi \wedge \exists x\psi) \to \exists x(\varphi \wedge \psi)$ 2 p

(b) $(\forall x\varphi \vee \forall x\psi) \to \forall x(\varphi \vee \psi)$ 2 p

Solution.

(a) This formula is *not* derivable for $\varphi = x \doteq y$ and $\psi = x \doteq z$, indeed soundness would then imply that the formula is true in every interpretation $\mathcal{A}, v$ but we have the following countermodel. Take $|\mathcal{A}| = \{a, b\}$ and v any valuation of the variables such that $v(y) = a$ and $v(z) = b$. On the one hand, the formula $(\exists x\, x \doteq y) \wedge (\exists x\, x \doteq z)$ holds in $\mathcal{A}, v$ because $\exists x\, x \doteq y$ holds in $\mathcal{A}, v$ since $[\![x \doteq y]\!]^{[x \mapsto v(y)]} = 1$, and similarly for $\exists x\, x \doteq z$. On the other hand, the formula $\exists x(x \doteq y \wedge x \doteq z)$ does not hold in $\mathcal{A}, v$ since that would imply $v(y) = v(z)$, which is not the case.

(b) Fix an interpretation $\mathcal{A}, v$. Suppose that $[\![\forall x\varphi \vee \forall x\psi]\!] = 1$, and assume without loss of generality that $[\![\forall x\varphi]\!] = 1$, which means that $[\![\varphi]\!]^{[x \mapsto a]} = 1$ for every $a \in |\mathcal{A}|$. It is then also true that $[\![\varphi \vee \psi]\!]^{[x \mapsto a]} = 1$ for every $a \in |\mathcal{A}|$, whence $[\![\forall x(\varphi \vee \psi)]\!] = 1$. This shows that $(\forall x\varphi \vee \forall x\psi) \to \forall x(\varphi \vee \psi)$ holds in every interpretation, hence that it is derivable in natural deduction by the completeness theorem.

11. Let Γ be a set of sentences and φ a sentence. Show that if $\Gamma \cup \{\varphi\}$ is inconsistent, then $\Gamma \vdash \neg\varphi$. 2 p

Solution. To say that $\Gamma \cup \{\varphi\}$ is inconsistent is to say that $\Gamma \cup \{\varphi\} \vdash \bot$. This means that there exists a derivation $\mathcal{D}$ with undischarged assumptions among $\Gamma \cup \{\varphi\}$ and which ends with the formula $\bot$. By adding an instance of the $\to I$ rule for the formula $\neg\varphi$ at the end of $\mathcal{D}$ which discharges the eventual assumption of φ, one obtains a derivation proving $\Gamma \vdash \neg\varphi$.

12.

(a) State the compactness theorem for predicate logic. 1 p

(b) Let $\mathcal{L} = (\langle ; \rangle)$ be the language with only equality. Prove that there does not exist a sentence φ in $\mathcal{L}$ such that φ is true in a structure if and only if that structure is finite. (You may take as given that there exists, for any number n, a sentence ψ_n saying that there are at least n elements). 3 p

(c) Conclude that there does not exist a sentence φ in $\mathcal{L}$ such that φ is true in a structure if and only if that structure is infinite. 1 p

Solution.

(a) The compactness theorem for predicate logic is the following. Let Γ be a theory, if every finite subset of Γ admits a model then Γ admits a model.

(b) Assume that such a formula φ exists. Consider the theory $\Gamma := \{\varphi\} \cup \{\psi_n \mid n \in \mathbb{N}\}$, any finite subset of Γ is contained in one of the sets $\Gamma_k := \{\varphi\} \cup \{\psi_n \mid n = 0, 1, \dots, k\}$. By definition of φ and ψ_n, any structure with at least k elements is a model of Γ_k. This ensures that every finite subset of Γ has a model, hence Γ has a model by compactness. But a model of Γ would need to be finite (to satisfy φ) and to have at least n elements for every $n \in \mathbb{N}$ (to satisfy each ψ_n). This is clearly impossible, hence such a formula φ cannot exist.

(c) If such a φ existed, then $\neg\varphi$ would hold in a structure precisely when that structure is not infinite, i.e, finite. This contradicts part (b), hence such a φ cannot exist.

13. In the language with only equality, it is the case that if a sentence is true in one infinite structure, then it is true in all infinite structures (you are not supposed to prove this!). Thus from 12 (c) we can conclude that there exists no sentence in the language of equality such that 1) there exists a structure in which the sentence is true, and 2) if the sentence is true in a structure then that structure must be infinite.

Show that if the language contains a unary function symbol, then this is no longer true. That is, show that in the language $\mathcal{L} = (\langle ; f \rangle)$ with only one unary function symbol, there exists a sentence φ such that 1) φ is true in some structure, and 2) for every structure, if φ is true in it then that structure must be infinite. (Hint: Consider the sentence you wrote in 6 (a) and the negation of the sentence you wrote in 6 (b)). 3 p

Solution. Recall the following fact: if X is a finite set (say of n elements) and $f : X \to X$ is injective then f is also surjective. Indeed, if f fails to be surjective then f maps the n elements of X into a set of at most $n-1$ elements and cannot be injective as a result.
We can rephrase this fact by saying that if $f : X \to X$ is an injective function which is not surjective then X has to be infinite. Let φ be the sentence displayed below.

$$(\forall x \forall y (f(x) \doteq f(y) \to x \doteq y)) \wedge \neg (\forall x \exists y \; x \doteq f(y))$$

A structure $\mathcal{A}$ satisfies φ precisely when $f^{\mathcal{A}} : |\mathcal{A}| \to |\mathcal{A}|$ is injective but not surjective. Hence, if φ is true in $\mathcal{A}$ then $|\mathcal{A}|$ is infinite, as desired.