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Matematiska institutionen
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Written Reexam/Skriftligt omprov
Logic 7.5 hp / Logik 7,5 hp
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No aids are allowed except paper, pen, and list of natural deduction rules. Write clearly and motivate your answers carefully. It is preferred that you write your answers in English, but Swedish is also allowed.

This exam consists of three parts. The Basic part has 9 problems (1–8) worth a total of 20 points. The Problem part has 4 problems (9–13) worth a total of 20 points. You can obtain a maximum of 40 points. A score of 15 or more points in the Basic part results in a passing grade.

Written Exam

Basic part

1. *Let C be a subset of the natural numbers $\mathbb{N}$ inductively defined by*

- $0 \in C$;
- *if* $n \in C$ *then* $n + 2 \in C$.

(a) *Explain in words what the set C consists of, that is to say, give a non-inductive definition of the set C.*

(b) *Give a proof by induction of your answer in (a).*

(c) *Give a non-recursive definition of the function $f : C \to \mathbb{N}$ defined recursively by*

$$\begin{aligned} f(0) &= 0 \\ f(n+2) &= f(n) + 1. \end{aligned}$$

4 p

2. *Let P_1 and P_2 be propositional variables, and let $\mathcal{A}$ be an interpretation such that $P_1^{\mathcal{A}}$ and $P_2^{\mathcal{A}}$ are both true.*

(a) *Compute the truth-value of $((\neg P_1) \vee P_2) \to \neg(P_1 \wedge P_2)$;* *2 p*

(b) *Based on your answer in (a), can you say whether the formula in (a) is valid or not? Why/why not?* *1 p*

3. *Give a natural deduction proof of $(\neg\psi \wedge (\phi \vee \psi)) \to \phi$.*

2 p

4. Let Γ be a set of propositional formulas and let ϕ be a propositional formula.

(a) What does $\Gamma \vdash \phi$ mean? What does $\Gamma \vDash \phi$ mean? State both definitions carefully. *2 p*

(b) State the soundness theorem and the completeness theorem for propositional logic. *1 p*

5. In the language with only equality ($\langle ; \rangle$) consider the sentence $\exists x \exists y \forall z(z = x \vee z = y)$. Give a structure $\mathcal{A}$ where this sentence is true. Give a structure $\mathcal{B}$ where the sentence is false. *2 p*

6. We think of the language with only one binary relation symbol ($\langle E; \rangle$) as the language of directed graphs, with $E(x, y)$ meaning that there is an edge from node x to node y.

- *A node a in a graph is said to have* out-degree one *if there is exactly one node to which there is an edge from a.*
- *A node a in a graph is said to have* in-degree one *if there is exactly one node from which there is an edge to a.*

Write a sentence in the language stating that every node has out-degree one, and a sentence stating that every node has in-degree one.

2 p

7. In the formula $\forall x_0 \exists x_2 \forall x_5 (R_1(x_0, x_1) \rightarrow R_2(f(x_3, x_2), x_0))$ which variables are free and which are bound? Is the term $f_2(x_6, x_5)$ free for x_1 in this formula?

2 p

8. What does it mean that a sentence in predicate logic is valid*? Prove carefully that the sentence $\exists x P_1(x)$ is not valid.*

2 p

Problem part

9. Provide derivations in natural deduction without any undischarged assumptions of the following formulas:

(a) $((\phi \rightarrow \delta) \wedge (\psi \rightarrow \sigma)) \rightarrow ((\phi \vee \psi) \rightarrow (\delta \vee \sigma))$ *2 p*

(b) $\exists y \forall x (P_1(x, y)) \rightarrow \forall x \exists y (P_1(x, y))$ *3 p*

10. For each of the following formulas decide, using a method of your choice, whether it is derivable in natural deduction (and justify your answer):

(a) $\forall x \exists y (P_1(x, y)) \rightarrow \exists y \forall x (P_1(x, y))$ *2 p*

(b) $\exists y \forall x (f(x) = y) \rightarrow \forall x (f(f(x)) = f(x))$ *2 p*

11. Let ($\langle E; \rangle$) be the language of graphs, as in Problem 7 above. Recall that a set of sentences is said to be independent *if no sentence is a logical consequence of the other sentences in the set. Show that the set of the following three sentences is independent (you may define simple structures by drawing diagrams with nodes and directed edges, as long as you are careful and do it clearly):*

1. $\forall x \forall y \forall z (E(x,y) \wedge E(y,z) \rightarrow E(x,z))$
2. $\forall x E(x,x)$
3. $\forall x \forall y (E(x,y) \rightarrow E(y,x))$

4 p

12. Let Γ be a set of sentences (in some fixed language). Let $\mathrm{Thry}(\Gamma)$ be the set of sentences ϕ such that $\Gamma \vdash \phi$. Show that for all sentences ψ, it is the case that $\Gamma \vdash \psi$ if and only if $\mathrm{Thry}(\Gamma) \vdash \psi$. *3 p*

13. The purpose of this exercise is to prove that the notion of connected graph is not first-order definable (the proof here is different from the one in class). Let ($\langle E; \rangle$) be the language of directed graphs, as in Problem 7 and Problem 12 above. Suppose that Γ is a set of sentences in this language such that for any structure $\mathcal{G}$ it is the case that $\mathcal{G} \vDash \Gamma$ if and only if $\mathcal{G}$ is a connected graph, in the sense that for all $a, b \in |\mathcal{G}|$ there exists a path from a to b, i.e. a sequence of elements $c_1, \ldots, c_n \in |\mathcal{G}|$ such that $\langle a, c_1 \rangle \in E^{\mathcal{G}}$, $\langle c_i, c_{i+1} \rangle \in E^{\mathcal{G}}$ for all $1 \leq i \leq n-1$, and $\langle c_n, b \rangle \in E^{\mathcal{G}}$.

(a) *Let ϕ be the sentence from Problem 7 stating that every node has out-degree one, and ψ be the sentence stating that every node has in-degree one (you can take both sentences as given, whether you wrote them out in 7 or not). Show that $\Gamma \bigcup \{\phi, \psi\}$ has models of arbitrary finite size n. (Hint: a connected graph where every node has in-degree and out-degree one is a cycle.)*

(b) *Show that for every natural number n there exists a sentence σ_n such that $\mathcal{G} \vDash \sigma_n$ if and only if $|\mathcal{G}|$ has at least n elements (you may write e.g. σ_3 explicitly and then just indicate how to do it for arbitrary n).*

(c) *Use the compactness theorem and (a) and (b) to show that $\Gamma \bigcup \{\phi, \psi\}$ must have an infinite model.*

As there are no infinite cycles, this concludes the proof. *4 p*

——–End of exam ——–