MM5024 VT25 Exam 2025-05-04 Solution

1 (a) $\nvDash (P_1 \to P_2) \to P_2$

A countermodel is given by $[\![P_1]\!] = 0$, $[\![P_2]\!] = 0$.

(b) $P_1 \vee (P_2 \to P_3) \nvDash (P_1 \vee P_2) \to (P_1 \wedge P_3)$

A countermodel is given by $[\![P_1]\!] = 1$, $[\![P_3]\!] = 0$, and $[\![P_2]\!]$ arbitrary.

(c) $P_1 \vee P_2 \nvDash ((P_1 \to P_3) \vee (P_2 \to P_3)) \to P_3$

A countermodel is given by $[\![P_1]\!] = 1$, $[\![P_2]\!] = 0$ and $[\![P_3]\!] = 0$.

2 (a) $\forall x_2 \exists x_3 (P_2(\underline{x_1}, x_2) \to x_2 \doteq f_2(\underline{x_1}, x_3))$

the only free variable is x_1.

(b) $((\forall x_1 \exists x_2 P_1(x_1, x_2)) \wedge P_2(f_1(\underline{x_1}, \underline{x_5})))$
$\to \forall x_1 (\exists x_2 P_1(x_1, x_2) \wedge P_2(f_1(x_1, \underline{x_5})))$

the free variables are x_1 and x_5

3 (a)

$$\cfrac{\cfrac{\neg P_1 \vee P_2 \qquad \cfrac{\cfrac{[\neg P_1]^2 \quad [P_1]^1}{\bot}}{P_2} \qquad [P_2]^3}{P_2}\,E_{\vee,2,3}}{P_1 \to P_2}\,I_{\to,1}$$

(b) We give two different derivations

$$
\begin{prooftree}
\AxiomC{$[\neg(\neg P_1 \vee P_2)]^1$}
\AxiomC{$[P_2]^2$}
\RightLabel{$I_{\vee,r}$}
\UnaryInfC{$\neg P_1 \vee P_2$}
\RightLabel{$E_{\to}$}
\BinaryInfC{$\perp$}
\RightLabel{$I_{\to,2}$}
\UnaryInfC{$\neg P_2$}
\AxiomC{$P_1 \to P_2$}
\AxiomC{$[\neg(\neg P_1 \vee P_2)]^1$}
\AxiomC{$[\neg P_1]^3$}
\RightLabel{$I_{\vee,l}$}
\UnaryInfC{$\neg P_1 \vee P_2$}
\RightLabel{$E_{\to}$}
\BinaryInfC{$\perp$}
\RightLabel{$RAA,3$}
\UnaryInfC{P_1}
\RightLabel{$E_{\to}$}
\BinaryInfC{P_2}
\RightLabel{$E_{\to}$}
\BinaryInfC{$\perp$}
\RightLabel{$RAA,1$}
\UnaryInfC{$\neg P_1 \vee P_2$}
\end{prooftree}
$$

$$
\begin{prooftree}
\AxiomC{$[\neg(\neg P_1 \vee P_2)]^1$}
\AxiomC{$[\neg(\neg P_1 \vee P_2)]^1$}
\AxiomC{$P_1 \to P_2$}
\AxiomC{$[P_1]^2$}
\RightLabel{$E_{\to}$}
\BinaryInfC{P_2}
\RightLabel{$I_{\vee,r}$}
\UnaryInfC{$\neg P_1 \vee P_2$}
\RightLabel{$E_{\to}$}
\BinaryInfC{$\perp$}
\RightLabel{$I_{\to,2}$}
\UnaryInfC{$\neg P_1$}
\RightLabel{$I_{\vee,l}$}
\UnaryInfC{$\neg P_1 \vee P_2$}
\RightLabel{$E_{\to}$}
\BinaryInfC{$\perp$}
\RightLabel{$RAA,1$}
\UnaryInfC{$\neg P_1 \vee P_2$}
\end{prooftree}
$$

14 see Carlström's course

5 We can take $\varphi_0 = P_1$ $\varphi_1 = P_2$ and $\varphi_2 = \neg(P_1 \wedge P_2)$

By soundness, to show theories are consistent it is enough to give a model.

- $\{\varphi_0, \varphi_1\}$ is consistent, a model is given by $[\![P_1]\!] = 1$, $[\![P_2]\!] = 1$
- $\{\varphi_0, \varphi_2\}$ is consistent, a model is given by $[\![P_1]\!] = 1$ $[\![P_2]\!] = 0$
- $\{\varphi_1, \varphi_2\}$ is consistent, a model is given by $[\![P_1]\!] = 0$ $[\![P_2]\!] = 1$

However $\{\varphi_0, \varphi_1, \varphi_2\}$ is not consistent

$$\dfrac{\neg(P_1 \wedge P_2) \qquad \dfrac{P_1 \quad P_2}{P_1 \wedge P_2}}{\bot}$$

shows that $\varphi_0, \varphi_1, \varphi_2 \vdash \bot$.

6 Let's suppose $\Gamma \vdash \varphi$, then

$$\dfrac{\neg\varphi \qquad \begin{array}{c}\Gamma \\ \vdots \\ \varphi\end{array}}{\bot}\,E_{\to}$$

shows that $\Gamma \cup \{\neg\varphi\} \vdash \bot$

Conversely, we suppose $\Gamma \cup \{\neg\varphi\} \vdash \bot$

then

$$\dfrac{\begin{array}{c}[\neg\varphi]^1 \quad \Gamma \\ \vdots \\ \bot\end{array}}{\varphi}\,RAA_1$$

where the RAA discharge all occurences of $\neg\varphi$

shows that $\Gamma \vdash \varphi$.

7 (a) We give a derivation,

$$\dfrac{\exists x_0 P_2(x_0) \qquad \dfrac{\dfrac{[P_2(x_0)]^1}{P_1(x_0) \to P_2(x_0)}\ I\!\to}{\exists x_0 (P_1(x_0) \to P_2(x_0))}\ I\exists}{\exists x_0 (P_1(x_0) \to P_2(x_0))}\ E\exists, 1$$

(b) We give a countermodel.

$|\mathcal{A}| = \{a, b\}$ $\quad [\![P_1(a)]\!] = 1 \quad [\![P_1(b)]\!] = 0$ and the valuation arbitrary.

$\quad [\![P_2(a)]\!] = 0 \quad [\![P_2(b)]\!] = 1$

then $[\![P_1(a) \vee P_2(a)]\!] = 1$ and $[\![P_1(b) \vee P_2(b)]\!] = 1$

implies $\mathcal{A} \models \forall x_0 (P_1(x_0) \vee P_2(x_0))$

but $\mathcal{A} \not\models \forall x_0 P_1(x_0)$ as $[\![P_1(b)]\!] = 0$

$\mathcal{A} \not\models \forall x_0 P_2(x_0)$ as $[\![P_2(a)]\!] = 0$

hence $\mathcal{A} \not\models (\forall x_0 P_1(x_0)) \vee (\forall x_0 P_2(x_0))$

By soundness, $\forall x_0 (P_1(x_0) \vee P_2(x_0)) \vdash (\forall x_0 P_1(x_0)) \vee (\forall x_0 P_2(x_0))$ cannot be derived.

8 By soundness it is enough to give a model.

We take $|\mathcal{A}| = \{a, b\}$ and the valuation arbitrary.
$$\begin{cases} [\![f_1(a)]\!] = b \text{ and } [\![f_1(b)]\!] = a \\ [\![P_1(a)]\!] = 1 \text{ and } [\![P_1(b)]\!] = 0 \end{cases}$$

- $[\![a = a \vee a = b]\!] = 1$ and $[\![b = a \vee b = b]\!] = 1$

  so $\mathcal{A} \models \exists x_1 \exists x_2 \forall x_3 (x_3 \doteq x_1 \vee x_3 \doteq x_2)$

- $[\![P_1(a) \to \neg P_1(f_1(a))]\!] = 1$ as $[\![\neg P_1(f_1(a))]\!] = [\![\neg P_1(b)]\!] = 1$

  $[\![P_1(b) \to \neg P_1(f_1(b))]\!] = 1$ as $[\![P_1(b)]\!] = 0$

  so $\mathcal{A} \models \forall x_1 (P_1(x_1) \to \neg P_1(f_1(x_1)))$

- $[\![P_1(a)]\!] = 1$, so $\mathcal{A} \models \exists x_1 \, P_1(x_1)$.

9

$$\dfrac{\exists x_0 P_1(x_0) \to \exists x_0 P_2(x_0) \qquad \dfrac{P_1(x_0)}{\exists x_0 P_1(x_0)}\, I\exists}{\exists x_0 P_2(x_0)}\, E\to \qquad \dfrac{\dfrac{[P_2(x_0)]^1}{P_1(x_0) \to P_2(x_0)}\, I\to}{\exists x_0 (P_1(x_0) \to P_2(x_0))}\, I\exists$$
$$\dfrac{\exists x_0 P_2(x_0) \qquad \exists x_0 (P_1(x_0) \to P_2(x_0))}{\exists x_0 (P_1(x_0) \to P_2(x_0))}\, E\exists,1$$

gives a derivation $\mathcal{D}_1$ of $\exists x_0 P_1(x_0) \to \exists x_0 P_2(x_0),\ P_1(x_0) \vdash \exists x_0 (P_1(x_0) \to P_2(x_0))$

$$\dfrac{\dfrac{\dfrac{\dfrac{\neg P_1(x_0) \qquad [P_1(x_0)]^1}{\bot}\, E\to}{P_2(x_0)}\, E\bot}{P_1(x_0) \to P_2(x_0)}\, I\to,1}{\exists x_0 (P_1(x_0) \to P_2(x_0))}\, I\exists$$

gives a derivation $\mathcal{D}_2$ of $\neg P_1(x_0) \vdash \exists x_0 (P_1(x_0) \to P_2(x_0))$

the full derivation is given by :

$$
\dfrac{
\dfrac{
\dfrac{
[\neg(P_1(x_0)\vee\neg P_1(x_0))]^1 \quad
\dfrac{
\dfrac{
\dfrac{
[\neg(P_1(x_0)\vee\neg P_1(x_0))]^1 \quad \dfrac{[P_1(x_0)]^2}{P_1(x_0)\vee\neg P_1(x_0)}\, I\vee, e
}{\bot}\, E\to
}{\neg P_1(x_0)}\, I\to, 2
}{P_1(x_0)\vee\neg P_1(x_0)}\, I\vee, 1
}{\bot}\, E\to
}{P_1(x_0)\vee\neg P_1(x_0)}\, RAA, 1
\qquad
\begin{array}{c}
\exists x_0 P_1(x_0)\to\exists x_1 P_2(x_0) \quad [P_1(x_0)] \\
\vdots\ \mathcal{D}_1 \\
\exists x_0(P_1(x_0)\to P_2(x_0))
\end{array}
\qquad
\begin{array}{c}
[\neg P_1(x_0)] \\
\vdots\ \mathcal{D}_2 \\
\exists x_0(P_1(x_0)\to P_2(x_0))
\end{array}
}{\exists x_0(P_1(x_0)\to P_2(x_0))}\, E\vee
$$

10 (a) True.

If $\Gamma_1\cap\Gamma_2 \vdash \bot$ then, as $\Gamma_1\cap\Gamma_2\subseteq\Gamma_1$, $\Gamma_1\vdash\bot$.

But as Γ_1 is consistent $\Gamma_1\nvdash\bot$ so $\Gamma_1\cap\Gamma_2\nvdash\bot$ and $\Gamma_1\cap\Gamma_2$ is consistent.

(b). False.

Take $\Gamma_1=\{P_1\}$ and $\Gamma_2=\{\neg P_1\}$,

$[\![P_1]\!]=1$ gives a model to Γ_1, and $[\![P_1]\!]=0$ gives a model to Γ_2 and by soundness Γ_1,Γ_2 are consistent.

But $\Gamma_1\cup\Gamma_2=\{P_1,\neg P_1\}$ is not consistent, $\dfrac{\neg P_1 \quad P_1}{\bot}\, I\to$ shows that $\Gamma_1\cup\Gamma_2\vdash\bot$.

(c) True.

If $\bigcup_{n\in\mathbb{N}} \Gamma_n$ is inconsistent, then a proof of $\perp$ with hypotheses in $\bigcup_{n\in\mathbb{N}} \Gamma_n$, only uses a finite number of hypotheses that we denote by $\varphi_0, \varphi_1, \dots, \varphi_k$.

Each φ_i belongs to some Γ_{n_i} and by taking $N := \max\{n_0, \dots, n_k\}$ each φ_i belongs to Γ_N (as $\Gamma_{n_i} \subseteq \Gamma_N$).

So we can derive $\perp$ from Γ_N and Γ_N is not consistent.

(d) False.

Let Γ_1, Γ_2 as in (b). As Γ_1 is consistent, we can find Γ_1^* a maximally consistent extension of Γ_1, and similarly Γ_2^* a maximally consistent extension of Γ_2.

But, as $\Gamma_1^* \cup \Gamma_2^* \supseteq \Gamma_1 \cup \Gamma_2$, $\Gamma_1^* \cup \Gamma_2^*$ is not consistent and so not maximally consistent.

$\boxed{11}$ We denote $t_0 = f_1$ and $t_{n+1} = f_2(t_n)$ for $n \in \mathbb{N}$

We can show by induction that $[\![t_n]\!]^{\mathcal{N},v} = n$ for all $n \in \mathbb{N}$.

Suppose $Th(\mathcal{N}, v) \vdash \exists x_0 \varphi$

then by soundness we have $\mathcal{N}, v \models \exists x_0 \varphi$

and so there exists some $n \in \mathbb{N}$ such that $\mathcal{N}, v[x_0 \mapsto n] \models \varphi$

and this implies $\mathcal{N}, v \models \varphi[t_n / x_0]$

As t_n is closed, this shows that $Th(\mathcal{N}, v)$ has the existence property.

12 (a) We let $t_1 := x_0$, and $t_{n+1} := f_2(x_0, t_n)$ for $n \geqslant 1$.

and we take $\varphi_n = P_1(t_n, f_1)$ for all $n \geqslant 1$.

(t_n represents $\overbrace{x_0 + \dots + x_0}^{n \text{ times}}$ so φ_n expresses " $n.x_0 < 1$ ")

(b) We shows that the theory has a model using compactness.
Let Γ_0 be a finite subtheory of $\mathcal{Th}(\mathbb{Q}) \cup \{\varphi_0, \varphi_1, \dots\}$
Then Γ_0 is contained in $\mathcal{Th}(\mathbb{Q}) \cup \{\varphi_0, \dots, \varphi_N\}$ for some $N > 0$.

It is thus enough to give a model of $\mathcal{Th}(\mathbb{Q}) \cup \{\varphi_0, \dots, \varphi_N\}$

We claim that $\mathbb{Q}, \nu[x_0 \mapsto \frac{1}{N+1}] \models \mathcal{Th}(\mathbb{Q}) \cup \{\varphi_0, \dots, \varphi_N\}$

$\mathbb{Q}, \nu[x_0 \mapsto \frac{1}{N+1}] \models \mathcal{Th}(\mathbb{Q})$ as the formulas in $\mathcal{Th}(\mathbb{Q})$ have no free variables
$\mathbb{Q}, \nu[x_0 \mapsto \frac{1}{N+1}] \models \varphi_0$ as $\frac{1}{N+1} > 0$
$\mathbb{Q}, \nu[x_0 \mapsto \frac{1}{N+1}] \models \{\varphi_1, \dots, \varphi_N\}$ as $\frac{1}{N+1} < \frac{1}{n}$ for $n = 1 \dots N$

So $\mathcal{Th}(\mathbb{Q}) \cup \{\varphi_0, \dots, \varphi_N\}$ has a model and Γ_0 also, as wanted.

(c) $\mathbb{Q}, \nu \models \varphi_0$ gives $\nu(x_0) > 0$
and for $n > 0$, $\mathbb{Q}, \nu \models \varphi_n$ gives $\nu(x_0) < \frac{1}{n}$

Hence we would have $\nu(x_0) > 0$ and $\nu(x_0) < \frac{1}{n}$ for all $n > 0$.

This is not possible as $\{x \in \mathbb{Q} \mid x > 0 \text{ and } \forall n > 0, x < \frac{1}{n}\} = \emptyset$.