MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Avd. Matematik
Examiner: Sven Raum
Instructor: Corentin Léna

Solutions to
Ordinary differential equations
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1. **Systems of differential equations** (20 points)
   Find the general solution of the system of differential equations

$$X' = AX$$

   where A is the matrix

$$\begin{pmatrix} 7 & 1 & \sqrt{2} \\ 1 & 7 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 6 \end{pmatrix}.$$

   Indication of solution. The matrix A is diagonalisable and after a calculation we find that its eigenvectors are

$$\begin{pmatrix} 1 \\ -1 \\ -\sqrt{2} \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ \frac{\sqrt{2}}{2} \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ 1 \\ -\frac{\sqrt{2}}{2} \end{pmatrix}.$$

   Their eigenvalues are 4,8 and 8, respectively. As a consequence the general solution to $X' = AX$ is given by

$$X = e^{t4} \begin{pmatrix} 1 \\ -1 \\ -\sqrt{2} \end{pmatrix} + e^{t8} \begin{pmatrix} 1 \\ 0 \\ \frac{\sqrt{2}}{2} \end{pmatrix} + e^{t8} \begin{pmatrix} 0 \\ 1 \\ -\frac{\sqrt{2}}{2} \end{pmatrix}.$$

2. **Higher order differential equations** (20 points)
   Solve the differential equation

$$f''(x) - f'(x) - 2f(x) = 6xe^{-x}$$
$$f(0) = 0$$
$$f'(0) = -\frac{2}{3}.$$

   Indication of solution. We find the general solution to the homogeneous equation $f''(x) - f'(x) - 2f(x) = 0$ and then use an well-chosen Ansatz to solve the inhomogeneous problem, possibly taking resonance cases into account.

   The characteristic polynomial associated with our homogeneous equation is $X^2 - X - 2X = (x+1)(x-2)$, so that a fundamental system of solutions for the homogeneous problem is

$$x \mapsto e^{-x}$$
$$x \mapsto e^{2x}.$$

   Based on the term $6xe^{-x}$ we make the Ansatz

$$f(x) = cxe^{-x} + dx^2e^{-x}.$$

   Taking derivatives of f and fitting constants, we find that indeed a solution to our equation can be found with the constants $c = -\frac{2}{3}$ and $d = -1$. The initial conditions are satisfied as well.

3. **Power series method** (20 points)
Solve the following differential equation by means of the power series method and express its solution as an elementary function.

$$-x(x+1)^2 f'(x) + (x+1)^2 f(x) = x^2$$
$$f(0) = 0$$
$$f(1) = \frac{1}{2}.$$

Indication of solution. We start by making the Ansatz that f is analytic with power series expansion

$$f(x) = \sum_{n \in \mathbb{N}} a_n x^n.$$

Then

$$f'(x) = \sum_{n \in \mathbb{N}} (n+1) a_{n+1} x^n.$$

Further the equation given in the question is equivalent to

$$-xf'(x) + f(x) = \left(\frac{x}{x+1}\right)^2. \tag{1}$$

We find the power series expansion of $(\frac{x}{x+1})^2$ by first considering $g(x) = \frac{1}{x+1}$. Taking a few derivatives, we guess that

$$g^{(n)}(x) = (-1)^n \cdot n! \cdot \frac{1}{x+1}^{n+1}$$

which is subsequently proven by induction. Consequently,

$$g(x) = \sum_{n \in \mathbb{N}} b_n x^n \qquad \text{with } b_n = \frac{g^{(n)}(0)}{n!} = (-1)^n.$$

Next note that $-g'(x) = \frac{1}{(x+1)^2}$, so that

$$\frac{1}{(x+1)^2} = \sum_{n \in \mathbb{N}} c_n \qquad \text{with } n! c_n = -(n+1)! b_{n+1}.$$

Explicitly, we find $c_n = (-1)^n (n+1)$. Putting this together with a degree shift, we find the right-hand side of (1).

$$\left(\frac{x}{x+1}\right)^2 = \sum_{n \in \mathbb{N}_{\geq 1}} (-1)^n (n-1) x^n.$$

Let us also explicitly describe the power series expansion of the left-hand side of (1).

$$-xf'(x) + f(x) = a_0 + \sum_{n \in \mathbb{N}_{\geq 1}} (1-n) a_n x^n.$$

Comparing coefficients we find that

$$a_0 = 0$$
$$(-1)^n (n-1) = (1-n) a_n \quad \text{for all } n \geq 1.$$

The second line can be simplified to $a_n = (-1)^{n+1}$ for $n \geq 2$. Note that for $n = 1$ there is no condition on a_1. So the power series expansion of f is

$$f(x) = a_1 x + \sum_{n \in \mathbb{N}_{\geq 2}} (-1)^{n+1} x^n.$$

We cannot directly make use of the initial condition $f(1)$, since the radius of convergence of the right hand side equals 1. So we first have to express f as a function. So let us write f is a more regular form first.

$$f(x) = (a_1 - 1)x + \sum_{n \in \mathbb{N}_{\geq 1}} (-1)^{n+1} x^n.$$

We recognise its expresseion as close to the one of $\frac{1}{x+1}$ and obtain

$$\frac{x}{x+1} = x \sum_{n \in \mathbb{N}} (-1)^n x^n = \sum_{n \in \mathbb{N}} (-1)^n x^{n+1} = f(x) - (a_1 - 1)x.$$

Substituting $x = 1$ as well as $f(1) = \frac{1}{2}$ in the first and last term of this equation, we find that $a_1 = 1$. So indeed $f(x) = \frac{x}{x+1}$. Note that the initial condition $f(0) = 0$ is automatically satisfied by this solution.

4. **Autonomous systems of differential equations** (20 points)
For the following autonomous system, find all equilibrium points and determine whether they are asymptotically stable, stable or unstable

$$\begin{Bmatrix} x' = \sin x \cdot \cos y \\ y' = x + y \end{Bmatrix}$$

Indication of solution. Writing

$$F(x, y) = \begin{pmatrix} \sin x \cdot \cos y \\ x + y \end{pmatrix},$$

the equilibrium points can be found by solving $F(x, y) = 0$. The solutions to this equation are exactly of the form

$$y = -x \text{ for } x \in \frac{\pi}{2}\mathbb{Z}.$$

Since all equilibrium points are isolated, we can use linearisation to understand their stability properties. We write

$$A = A(x, y) = \begin{pmatrix} \cos x \cdot \cos y & -\sin x \cdot \sin y \\ 1 & 1 \end{pmatrix}$$

for the Jacobian of F. We have to find the eigenvalues of A. We have

$$\det A = \cos x \cos y + \sin x \sin y$$

which simplifies under the assumption $y = -x$ to

$$\cos^2 x - \sin^2 x = 1 + 2\sin^2 x.$$

We will next plug in values $x \in \frac{\pi}{2}\mathbb{Z}$. Here we distinguish two cases. If $x \in \pi\mathbb{Z}$, then

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

and thus $\det A = 1$ and $\mathrm{Tr} A = 2$, telling us that $(x, -x)$ is an unstable equilibrium point. If $x \in \pi\mathbb{Z} + \frac{\pi}{2}$, then

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

and thus $\det A = -1$ and $\mathrm{Tr} = 1$, telling us that also here $(x, -x)$ is an unstable equilibrium point.

5. **Boundary value problems** (20 points)
Consider the differential equation

$$u'' - 2xu' + 2nu = 0 \tag{$*$}$$

for a parameter $n \in \mathbb{N}$.

(a) Rewrite the differential equation in Sturm-Liouville form.

(b) Find a solution H_0 for the boundary value problem

$$\begin{aligned} u'' - 2xu' &= 0, \\ H_0'(0) = 0 &= H_0'(1). \end{aligned}$$

(c) Show that if H_n is a solution of $(*)$ for the parameters n, then there is a solution H_{n-1} for the parameter $n-1$ that satisfies $H_n' = nH_{n-1}$.

(d) Use the statement of the previous item to find solutions H_1, H_2, H_3, H_4 for the differential equation $(*)$ with parameters $n = 1, 2, 3, 4$.

Indication of solution.

(a) The SL-form of this equation is $(e^{-x^2}u')' + 2ne^{-x^2}u = 0$.

(b) The constant function $u \equiv 1$ is a solution.

(c) If H_n is a solution of $(*)$ for the parameter $n \geq 1$, then we have to check that

$$H_{n-1} = \frac{1}{n}H_n'$$

defines a solution for the equation with parameter $n-1$. We use the fact that

$$H_n'' = 2xH_n' - 2nH_n$$

combined with the product rule, to find that

$$\begin{aligned} H_{n-1}'' - 2xH_{n-1}' + 2(n-1)H_{n-1} &= \frac{1}{n}H_n''' - \frac{1}{n}2xH_n'' + 2(n-1)\frac{1}{n}H_n' \\ &= \frac{1}{n}(2xH_n'' + 2H_n') - \frac{1}{n}2nH_n' - \frac{1}{n}2xH_n'' + \frac{2(n-1)}{n}H_n' \quad = 0. \end{aligned}$$

(d) We will find the asked solutions recursively. From the previous item, our task to find H_n from H_{n-1} consists in fixing a constant in the primitive equation

$$H_n = n\int H_{n-1} + C.$$

Recall that H_0 is a solution to $(*)$ for the parameter $n = 0$. Thus

$$H_1 = 1 \cdot \int 1\mathrm{d}x + C = x + C$$

for some $C \in \mathbb{R}$. Calculating $H_1'(x) = C$, $H_1''(x) = 0$ and plugging these into $(*)$, we find

$$0 = 0 - 2x \cdot 1 + 2(x + C).$$

This implies $C = 0$ and thus $H_1(x) = x$. Similarly we find that

$$\begin{aligned} H_2(x) &= x^2 - \frac{1}{2} \\ H_3(x) &= x^3 - \frac{3}{2}x \\ H_4(x) &= x^4 - 3x^2 + \frac{3}{4}. \end{aligned}$$