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Solutions to
Ordinary differential equations
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1. **Systems of differential equations** (20 points)
Find the general solution of the system of differential equations

$$X' = AX$$

where A is the matrix

$$\begin{pmatrix} -2 & 1 & -1 \\ -3 & 2 & -1 \\ -1 & 1 & 0 \end{pmatrix}.$$

Idea of solution. The matrix A is diagonalisable and after a calculation we find that its eigenvectors are

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}.$$

Their eigenvalues are -1, 1 and 0, respectively. As a consequence the general solution to $X' = AX$ is given by

$$X = ae^{-t} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + be^t \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + c \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}.$$

2. **Higher order differential equations** (20 points)
Solve the differential equation with boundary conditions

$$f''(x) + f'(x) - 2f(x) = 3e^x - 18xe^{-2x}$$
$$f(0) = 0$$
$$f'(-1) = -6e^2.$$

Idea of solution. We find the general solution to the homogeneous equation $f''(x)+f'(x)-2f(x) = 0$ and then use a well-chosen Ansatz to solve the inhomogeneous problem, possibly taking resonance cases into account.

The characteristic polynomial associated with our homogeneous equation is $X^2 + X - 2X = (X + 2)(X - 1)$, so that a fundamental system of solutions for the homogeneous problem is

$$x \mapsto e^{-2x}$$
$$x \mapsto e^x.$$

Based on the terms xe^x and xe^{-2x} we make three Ansatz, whose (suitable) linear combination will yield a solution to our problem.

$$f_1(x) = axe^x$$
$$f_2(x) = bxe^{-2x}$$
$$f_3(x) = cx^2e^{-2x}.$$

We find that

$$\begin{aligned} f_1''(x) + f_1'(x) - 2f_1(x) &= 3ae^x \\ f_2''(x) + f_2'(x) - 2f_2(x) &= -3be^{-2x} \\ f_3''(x) + f_3'(x) - 2f_3(x) &= c(2-6x)e^{-2x}. \end{aligned}$$

If $f_1 + f_2 + f_3$ solves the inhomogenous problem, then $a = 1$ and $2c = 3b$ must hold. Fitting the boundary conditions, we find

$$a = 1 \quad b = 2 \quad c = 3.$$

3. **Laplace transform** (20 points)
Solve the following initial value problem by means of the Laplace transform and express its solution as an elementary function.

$$\begin{aligned} f''(x) + f(x) &= x^2 \\ f(0) &= 0 \\ f'(0) &= 0. \end{aligned}$$

Idea of solution. We apply the Laplace transform to the DE $f''(x) + f(x) = x^2$ and obtain

$$\mathrm{L}[f'' + f](p) = \mathrm{L}[x^2](p)$$

which yields after simplication

$$(p^2 + 1)\mathrm{L}[f](p) = \frac{2}{p^3}.$$

Solving for $\mathrm{L}[f]$ and using partial faction decomposition, we find that

$$\mathrm{L}[f](p) = \frac{-2}{p} + \frac{2}{p^3} + \frac{2p}{p^2+1}.$$

This is recognised as the Laplace transform of

$$-2 + x^2 + 2\cos(x).$$

A short calculation shows that this function indeed solves the initial value problem.

4. **Autonomous systems of differential equations** (20 points)
For the following autonomous system, find all equilibrium points and determine whether they are asymptotically stable, stable or unstable

$$\begin{Bmatrix} x' = & \frac{1}{2}\sin^2(x) + y \\ y' = & x^2 - y \end{Bmatrix}$$

Idea of solution. Writing

$$F(x, y) = \begin{pmatrix} \frac{1}{2}\sin^2(x) + y \\ x^2 - y \end{pmatrix},$$

the equilibrium points can be found by solving $F(x, y) = 0$. There is a unique solution, which is $(x, y) = (0, 0)$.

Since this equilibrium is isolated, we try to use linear approximation to determine its stability properties. The Jacobian of F is

$$\begin{pmatrix} \cos^2(x) sin(x) & 1 \\ 2x & -1 \end{pmatrix}$$

which at $(0,0)$ takes the value

$$\begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}.$$

The eigenvalues of this matrix are 0 and -1, so that linear approximation does not give any affirmative answer to the stability question. But $(1,0)^t$ being in the kernel of the Jacobian, points towards the investigation of initial conditions $(x_0, 0)$ with $x_0 > 0$. We observe that with these initial conditions, a trajectory $(x(t), y(t))$ has non-negative y-coordinate. Further, $x'(t) \geq 0$ for all t and $x' > 0$ as long as $x(t) \leq \pi$. We conclude that $(0,0)$ is an unstable equilibrium point.

5. **Boundary value problems** (20 points)
Show that for any $\lambda > 0$ the following boundary value problem has a unique solution. Express this solution as an elementary function.

$$\begin{aligned} u'' &= \lambda u \text{ on } [0,1] \\ u(0) &= 0 \\ u'(1) &= 1. \end{aligned}$$

Idea of solution. We solve the second order linear DE $u'' - \lambda u = 0$ and obtain the fundamental system of solutions

$$\begin{aligned} x &\mapsto e^{\lambda^{1/2} x} \\ x &\mapsto e^{-\lambda^{1/2} x}. \end{aligned}$$

So the general solution of this DE is

$$u(x) = a e^{\lambda^{1/2} x} + b e^{-\lambda^{1/2} x}.$$

The boundary condition $u(0) = 0$ implies that $a = -b$, that is the general solution satisfying $u(0) = 0$ is of the form

$$u(x) = c(e^{\lambda^{1/2} x} - e^{-\lambda^{1/2} x}) = 2c \sinh(\lambda^{1/2} x).$$

Calculating the derivative of this function we find

$$u'(x) = 2c\lambda^{1/2} \cosh(\lambda^{1/2} x).$$

The boundary condition $u'(1) = 1$ now fixes, since $\lambda \neq 0$ holds,

$$c = \frac{1}{2\lambda^{1/2} \cosh(\lambda^{1/2})}.$$

So we find the unique solution

$$u_\lambda(x) = \frac{1}{\lambda^{1/2}} \frac{\cosh(\lambda^{1/2} x)}{\cosh(\lambda^{1/2})}.$$