MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Avd. Matematik
Examinator: Jonathan Rohleder

Written examination in
Mathematics III – ODE
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1. **(4p)** Solve the initial value problem $y' = xy^2 + x$, $y(0) = 1$.

   **Solution:** The DE may be rewritten $y' = x(y^2 + 1)$ and, hence, has separate variables. Note that the $y$-dependent factor never gets zero. Dividing by $y^2 + 1$ and integrating yields

$$\int \frac{1}{1+y^2} dy = \int x dx,$$

   equivalently $\arctan y = \frac{1}{2}x^2 + C$ with $C \in \mathbb{R}$. Thus $y = \tan(\frac{x^2}{2} + C)$ is the general solution to the DE. To satisfy the initial condition we need

$$1 = y(0) = \tan C = \frac{\sin C}{\cos C},$$

   which has $C = \pi/4$ as a solution. Thus the unique solution to the BVP is

$$y = \tan\left(\frac{x^2}{2} + \frac{\pi}{4}\right).$$

2. **(6p)** Let $a \in \{1, \ldots, 12\}$ be the number of your month of birth. (For instance, $a = 1$ if you are born in January, $a = 7$ if you are born in July, or $a = 10$ if you are born in October.) For your $a$, determine the general solution to the system

$$\begin{cases} x' = -x + y, \\ y' = -x - 3y, \\ z' = -x - (a+3)y + az. \end{cases}$$

   **Solution:** We sketch a possible solution in dependence of the parameter $a$. The system is homogeneous and may be written

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = A \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \text{where} \quad A = \begin{pmatrix} -1 & 1 & 0 \\ -1 & -3 & 0 \\ -1 & -(a+3) & a \end{pmatrix}.$$

   The characteristic polynomial of $A$ is $p_A(\lambda) = (\lambda + 2)^2(a - \lambda)$, which leads to the eigenvalues $-2$ (with algebraic multiplicity 2) and $a$ with multiplicity one. Computing the corresponding eigenvectors yields $(-1, 1, 1)^\top$ for $\lambda = -2$ and $(0, 0, 1)^\top$ for $\lambda = a$. In particular, $A$ is not diagonalizable. In order to find a block-diagonalization, one may solve the linear system of equations $(A - (-2))v = (-1, 1, 1)^\top$. If $v$ is a solution then $v$ together with $(-1, 1, 1)^\top$ form a basis of $\ker(A - (-2))^2$. The equations to solve read

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & -1 \\ -1 & -1 & 0 & 1 \\ -1 & -(a+3) & a+2 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -(a+2) & a+2 & 0 \end{array}\right)$$

   by Gauss elimination. Thus $v = (-2, 1, 1)^\top$ is a solution. Set

$$T := \begin{pmatrix} -1 & -2 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix},$$

formed out of the eigenvectors and $v$ as its columns. We compute

$$T^{-1}AT = \begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & a \end{pmatrix} =: D,$$

a block diagonalization. Now we can compute $e^{tA}$. For the upper block we get

$$e^{t\begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix}} = e^{\begin{pmatrix} -2t & 0 \\ 0 & -2t \end{pmatrix}} e^{\begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}} = \begin{pmatrix} e^{-2t} & 0 \\ 0 & e^{-2t} \end{pmatrix} \left( I + \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} e^{-2t} & te^{-2t} \\ 0 & e^{-2t} \end{pmatrix}.$$

Therefore

$$e^{tA} = Te^{tD}T^{-1} = T \begin{pmatrix} e^{-2t} & te^{-2t} & 0 \\ 0 & e^{-2t} & 0 \\ 0 & 0 & e^{at} \end{pmatrix} T^{-1} = \begin{pmatrix} te^{-2t} + e^{-2t} & te^{-2t} & 0 \\ -te^{-2t} & e^{-2t} - te^{-2t} & 0 \\ -te^{-2t} & -e^{at} - te^{-2t} + e^{-2t} & e^{at} \end{pmatrix},$$

and we end up with the general solution

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}(t) = c_1 \begin{pmatrix} te^{-2t} + e^{-2t} \\ -te^{-2t} \\ -te^{-2t} \end{pmatrix} + c_2 \begin{pmatrix} te^{-2t} \\ e^{-2t} - te^{-2t} \\ -e^{at} - te^{-2t} + e^{-2t} \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ e^{at} \end{pmatrix}$$

with $c_1, c_2, c_3 \in \mathbb{R}$.

3. **(4p)** Consider the initial value problem

$$y' = 2x + y + 1, \quad y(1) = 1.$$

Compute a numerical solution at $x = 3$ by using the (forward) Euler method with step length $h = 1/2$.
**Solution:** We have $x_0 = 1, y_0 = 1$. As $h = 1/2$ we get

$$\begin{aligned} x_1 &= x_0 + 1/2 = 3/2, \\ y_1 &= y_0 + \frac{1}{2}(2x_0 + y_0 + 1) = 1 + 4/2 = 3, \\ x_2 &= x_1 + 1/2 = 2, \\ y_2 &= y_1 + \frac{1}{2}(2x_1 + y_1^2 + 1) = 3 + 7/2 = 13/2, \\ x_3 &= 5/2, \\ y_3 &= 13/2 + \frac{1}{2}(4 + 13/2 + 1) = 49/4, \\ x_4 &= 3, \\ y_4 &= 49/4 + \frac{1}{2}(5 + 49/4 + 1) = 171/8. \end{aligned}$$

As $x_4 = 3$, $y_4 = 171/8$ is the numerical approximation after four Euler steps of length $h = 1/2$ of $y(3)$.

4. **(4p)** Show that for each $x_0 \in \mathbb{R}$ the initial value problem

$$y' = \frac{3|y|\cos x}{2 + x^2}, \quad y(x_0) = 0,$$

has a unique solution defined on all of $\mathbb{R}$.

**Solution:** Consider the infinite strip $[x_0-\alpha, x_0+\alpha]\times\mathbb{R}$ for $\alpha \in \mathbb{R}$. The right-hand side $f(x,y) = \frac{3|y|\cos x}{2+x^2}$ is continuous on the whole strip as its numerator and denominator are continuous functions and the denominator never gets zero. Moreover, it satisfies a Lipschitz condition w.r.t. $y$ in the strip since

$$|f(x,y_1) - f(x,y_2)| = \frac{3|\cos x|}{2+x^2}\big||y_1| - |y_2|\big| \leq \frac{3}{2}|y_1 - y_2|.$$

This guarantees a unique solution defined on the whole interval $[x_0-\alpha, x_0+\alpha]$. As this is true for each $\alpha$ and the Lipschitz constant is independent of $\alpha$ (we may choose $L = \frac{3}{2}$), it follows that the solution is defined on all of $\mathbb{R}$. (Of course this unique solution is the constant zero function.)

5. **(6p)** Let again $a$ be the number of your month of birth. Determine all equilibrium points of the autonomous system

$$\begin{cases} \frac{dx}{dt} = x^2 - y^2, \\ \frac{dy}{dt} = (-1)^a(a+1)x + y - 1, \end{cases}$$

and investigate whether these equilibrium points are asymptotically stable.

**Solution:** We again provide a sketch of a solution in dependence of the parameter $a$. We distinguish two cases.

<u>$a$ even:</u> Here we have

$$f(x,y) = \begin{pmatrix} x^2 - y^2 \\ (a+1)x + y - 1 \end{pmatrix}, \quad f'(x,y) = \begin{pmatrix} 2x & -2y \\ a+1 & 1 \end{pmatrix}.$$

The equations for equilibrium points are thus $x^2 = y^2$ and $(a+1)x + y - 1 = 0$. The first one gives $y = \pm x$. If $y = x$ then from the second equation we get $(a+2)x = 1$, i.e. $x = 1/(a+2)$. On the other hand, if $y = -x$ then we obtain $x = 1/a$. Thus we have equilibrium points

$$\left(\frac{1}{a+2}, \frac{1}{a+2}\right), \quad \left(\frac{1}{a}, -\frac{1}{a}\right).$$

Stability: By linearization. The matrix

$$f'\left(\frac{1}{a+2}, \frac{1}{a+2}\right) = \begin{pmatrix} 2/(a+2) & -2/(a+2) \\ a+1 & 1 \end{pmatrix}$$

has eigenvalues

$$\frac{1}{2(a+2)}\left(a + 4 \pm \sqrt{-7a^2 - 24a - 16}\right).$$

The term under the square root gets negative for all natural numbers $a$ and therefore the eigenvalues are non-real with real part $(a+4)/(2(a+2))$, which is positive. Hence $(1/(a+2), 1/(a+2))$ is **unstable**. For the other point we have

$$f'\left(\frac{1}{a}, -\frac{1}{a}\right) = \begin{pmatrix} 2/a & 2/a \\ a+1 & 1 \end{pmatrix}$$

with eigenvalues

$$\frac{1}{2a}\left(a + 2 \pm \sqrt{9a^2 + 4a + 4}\right).$$

Here the eigenvalues are real but at least the one with $+$ is positive. Hence also $(1/a, -1/a)$ is **unstable**.

a odd: Here we have

$$f(x,y) = \begin{pmatrix} x^2 - y^2 \\ -(a+1)x + y - 1 \end{pmatrix}, \quad f'(x,y) = \begin{pmatrix} 2x & -2y \\ -(a+1) & 1 \end{pmatrix}.$$

The equations for equilibrium points are thus $x^2 = y^2$ and $-(a+1)x + y - 1 = 0$. The first one gives $y = \pm x$. If $y = x$ then from the second equation we get $-ax = 1$, i.e. $x = -1/a$. On the other hand, if $y = -x$ then we obtain $x = -1/(a+2)$. Thus we have equilibrium points

$$\left(-\frac{1}{a+2}, \frac{1}{a+2}\right), \quad \left(-\frac{1}{a}, -\frac{1}{a}\right).$$

Stability: By linearization. The matrix

$$f'\left(-\frac{1}{a+2}, \frac{1}{a+2}\right) = \begin{pmatrix} -2/(a+2) & -2/(a+2) \\ -(a+1) & 1 \end{pmatrix}$$

has eigenvalues

$$\frac{1}{2(a+2)}\left(a \pm \sqrt{9a^2 + 32a + 32}\right).$$

This is always real and at least the solution with + is positive, thus $(-1/(a+2), 1/(a+2))$ is **unstable**. For the other point we have

$$f'\left(-\frac{1}{a}, -\frac{1}{a}\right) = \begin{pmatrix} -2/a & 2/a \\ -(a+1) & 1 \end{pmatrix}$$

with eigenvalues

$$\frac{1}{2a}\left(a - 2 \pm \sqrt{-7a^2 - 4a + 4}\right).$$

The term under the square root is negative for each integer $a$, hence these are non-real eigenvalues with real part $(a-2)/(2a)$. This real part is negative if $a = 1$, otherwise positive. Thus **if** $a = 1$ then $(-1/a, -1/a)$ is **asymptotically stable**. For **all other odd** $a$, $(-1/a, -1/a)$ is **unstable**.

6. **(6p)** Let again $a$ be the number of your month of birth. Consider the boundary value problem

$$2y'' - ay' = f(x) \text{ on } [0,1], \qquad y(0) = c_0, \quad y(1) = c_1. \tag{1}$$

(a) Prove that for each $f \in \mathcal{C}[0,1]$ and all $c_0, c_1 \in \mathbb{R}$ the problem (1) has a unique solution.

(b) Solve the problem (1) for $f(x) = e^{ax}$, $c_0 = 0$ and $c_1 = e^a/a^2$.

**Solution:** Again we show a parameter-dependent solution.

(a) The characteristic polynomial of the homogeneous equation equals $p(\lambda) = 2\lambda^2 - a\lambda = \lambda(2\lambda - a)$ and has its roots at $\lambda = 0$ and $\lambda = a/2$. Thus the general solution of the homogeneous DE equals $y(x) = \alpha + \beta e^{ax/2}$ with $\alpha, \beta \in \mathbb{R}$. By Theorem 1 on p. 178 in the course book it suffices to show that the homogeneous DE with homogeneous boundary conditions $y(0) = y(1) = 0$ is uniquely solvable. In fact,

$$0 = y(0) = \alpha + \beta,$$
$$0 = y(1) = \alpha + \beta e^{a/2}$$

is the homogeneous linear system of equations described by the matrix

$$\begin{pmatrix} 1 & 1 \\ 1 & e^{a/2} \end{pmatrix},$$

which has determinant $e^{a/2} - 1 \neq 0$. Hence the homogeneous BVP has only the trivial solution, which implies assertion (a).

(b) We try an ansatz of the form $y_{\mathrm{p}}(x) = \gamma e^{ax}$. This satisfies the inhomogeneous DE if

$$e^{ax} = 2y_{\mathrm{p}}''(x) - ay_{\mathrm{p}}'(x) = \gamma\left(2a^2 - a^2\right)e^{ax} = \gamma a^2 e^{ax}.$$

This is satisfied if $\gamma = 1/a^2$. Hence $y_{\mathrm{p}}(x) = e^{ax}/a^2$, and therefore the general solution to the inhomogeneous DE is

$$y(x) = \alpha + \beta e^{ax/2} + \frac{e^{ax}}{a^2}.$$

We determine $\alpha$ and $\beta$ in order to satisfy the boundary conditions:

$$\begin{aligned} 0 = y(0) &= \alpha + \beta + \frac{1}{a^2}, \\ \frac{e^a}{a^2} = y(1) &= \alpha + \beta e^{a/2} + \frac{e^a}{a^2} \end{aligned}$$

has the solutions

$$\alpha = -\frac{e^{a/2}}{a^2(e^{a/2}-1)}, \quad \beta = \frac{1}{a^2(e^{a/2}-1)}.$$

Hence the solution to the boundary value problem is

$$y(x) = -\frac{e^{a/2}}{a^2(e^{a/2}-1)} + \frac{1}{a^2(e^{a/2}-1)}e^{ax/2} + \frac{e^{ax}}{a^2}.$$