MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Avdelning Matematik
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Solution to the exam in
ODE, 7.5 hp
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1 We solve the homogeneous equation first:

$$\lambda^2 - 5\lambda + 6 = 0 \Rightarrow \lambda_{1,2} = 2, 3$$
$$\Rightarrow y_{hom} = c_1 e^{2x} + c_2 e^{3x}.$$

Particular solution to the non-homogeneous equation can be found in the form

$$y_p = (ax+b)e^x.$$

Substitution into the original equation gives:

$$2ae^x + (ax+b)e^x - 5\big(ae^x + (ax+b)e^x\big) + 6(ax+b)e^x = 2xe^t.$$

Comparing the coefficients in front of xe^x and e^x we get the linear system:

$$\begin{cases} a - 5a + 6a = 2, \\ 2a + b - 5a - 5b + 6b = 0 \end{cases} \Rightarrow a = 1, b = 3/2.$$

The general solution to the differential equation is

$$y = (x + 3/2)e^x + c_1 e^{2x} + c_2 e^{3x}.$$

To satisfy the initial conditions we need to choose c_1 and c_2:

$$\begin{cases} 3/2 + c_1 + c_2 = 0 \\ 1 + 3/2 + 2c_1 + 3c_2 = 0 \end{cases} \Rightarrow c_1 = -2, c_2 = 1/2.$$

Summing up the solution is

$$y = (x + \frac{3}{2})e^t - 2e^{2t} + \frac{1}{2}e^{3t}.$$

2 Assume that the solution is given by the power series: $y = \sum_{k=0}^{\infty} a_k x^k$. Then we have:

$$\begin{array}{l} y = a_0 + a_1 x + a_2 x^2 + \dots, \\ y' = a_1 + 2a_2 x + 3a_3 x^3 + \dots, \\ y'' = 2a_2 + 6a_3 x + \dots \end{array}$$

Substitution into the differential equation gives:

$$x^2\Big(2a_2 + 6a_3 x + \dots\Big) + x\Big(a_1 + 2a_2 x + 3a_3 x^3 + \dots\Big) - \Big(a_0 + a_1 x + a_2 x^2 + \dots\Big) = x + x^2 + x^3 + \dots$$

We check coefficients in front of different powers of x:

$$\begin{array}{ll} x^0: & -a_0 = 0 \\ x^1: & a_1 - a_1 = 1 \\ x^2: & 2a_2 + 2a_2 - a_2 = 1 \end{array}$$

Already the second equality is impossible to satisfy implying that no solutions given by the power series exists.

3 The matrix

$$\begin{pmatrix} -1 & -7 \\ -7 & -1 \end{pmatrix}$$

has eigenvalues 6 and -8 with the eigenvectors $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, respectively. General solution to the linear system is given by

$$c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{6t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-8t}.$$

The solution satisfying the initial conditions is

$$3 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-8t}.$$

The system is unstable since one of the eigenvalues is positive ($\lambda = 6$).

4 The equilibrium points must satisfy

$$\begin{aligned} -2\sin x + (x+2y)^2 = 0 \\ -(x+2y) = 0 \end{aligned}$$

implying in particular that $\sin x = 0 \Rightarrow x = \pi n, n \in \mathbb{Z}$. Then y is calculated form the second equation $y = -\frac{\pi}{2} n$.

The linearised system near $(0,0)$ is

$$\begin{cases} \frac{dx}{dt} & = & -2x, \\ \frac{dy}{dt} & = & -x - 2y. \end{cases}$$

The Liapunov function can be taken in the form $L(x,y) = x^2 + y^2 > 0, \quad (x,y) \neq (0,0)$. Really we have

$$L_x x' + L_y y' = 2x(-2x) + 2y(-x-2y) = -4x^2 - 2xy - 4y^2 = -3x^2 - (x+y)^2 - 3y^2 < 0, \quad (x,y) \neq (0,0).$$

5 The operator os symmetric:

$$\begin{aligned} < Lu, v > &= \textstyle\int_0^\pi (-u'')v dx = -u'(x)v(x)|_{x=0}^\pi + u(x)v'(x)|_{x=0}^\pi + \int_0^\pi u(-v'')dx \\ &= -\underbrace{u'(\pi)}_{=0} v(\pi) + u'(0)\underbrace{v(0)}_{=0} + u(\pi)\underbrace{v'(\pi)}_{=0} - \underbrace{u(0)}_{=0} v'(0) + < u, Lv >, \end{aligned}$$

since u, v belong to the domain and satisfy the boundary condiitions.

The eigenfunctions are solution to the equation $y'' = \lambda y, \quad \lambda = k^2$. Every solution is of the form

$$y = a \sin kx + b \cos kx.$$

Checking boundary condition at $x = 0$ we conclude that

$$y = a \sin kx.$$

The boundary condition at $x = \pi$ gives

$$\cos k\pi = 0 \Rightarrow k = 1/2 + n, \quad n \in \mathbb{N}.$$

Hence the eigenvalues and the eigenfunctions are $\lambda_n = \left(\frac{1}{2} + n\right)^2$, $\psi_n = a_n \sin(\frac{1}{2} + n)x$, $n = 0, 1, 2, 3, \ldots$ The normalisation constants a_n are calculated from

$$(a_n)^{-2} = \int_0^\pi \left(\sin(\frac{1}{2} + n)x\right)^2 dx = \pi/2 \Rightarrow a_n = \sqrt{\frac{2}{\pi}}.$$

We get the following eigenfunction expansion:

$$f(x) = \sum_{n=0}^\infty c_n \sqrt{\frac{2}{\pi}} \sin(\frac{1}{2} + n)x, \quad c_n = \int_0^\pi f(x) \sqrt{\frac{2}{\pi}} \sin(\frac{1}{2} + n)x dx.$$