MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Avd. Matematik
Examinator: Annemarie Luger

Written exam in
Ordinary differential equations,
May 23, 2024

No calculator, book or notes are allowed. Give complete justifications for your answers!
At least 14 points (including bonus) are needed in order to proceed to the ***voluntary oral exam!***

1. Determine the solution(s) of the differential equation

$$xx' = t(x^2 + 1)$$

satisfying the initial condition:

(a) $x(0) = 1$
(b) $x(0) = -1$
(c) $x(0) = 0$.

Comment on your result with respect to the *existence and uniqueness theorem*! (4p)

2. Is there an *ordinary linear homogeneous second order differential equation with constant coefficients* for which the functions $x_1(t) = \cos 2t$ and $x_2(t) = t^2 e^{-7t}$ are solutions? If yes, give an example. If no, explain why! (2p)

3. Use Laplace transform to solve the initial value problem (4p)

$$x'' - 2x' = -2e^{2t}, \quad \text{with} \quad x(0) = 1, \quad x'(0) = -1.$$

On the backside of this sheet you find a list of some frequently occurring Laplace transforms.

4. (a) Determine a fundamental matrix for the homogeneous system (2p)

$$\vec{x}\,' = \begin{pmatrix} -1 & 3 \\ 0 & 1 \end{pmatrix} \vec{x}.$$

(b) Give the general solution of the inhomogeneous system (4p)

$$\vec{x}\,' = \begin{pmatrix} -1 & 3 \\ 0 & 1 \end{pmatrix} \vec{x} + \begin{pmatrix} 0 \\ e^{2t} \end{pmatrix}.$$

5. Consider the following boundary value problem (BVP)

$$\begin{aligned} -x'' &= \lambda x \\ x(0) &= x'(0) \\ x(1) &= x'(1). \end{aligned}$$

(a) For which values of $\lambda \in \mathbb{R}$ does the BVP have a non-trivial solution? Determine also these solutions. (3p)

(b) Are there values of $\lambda \in \mathbb{C} \setminus \mathbb{R}$ for which the BVP has a non-trivial solution? Motivate your answer! (1p)

Please turn the page!

6. Consider the system

$$\dot{x} = -y + x\cos(x^2 + y^2)$$
$$\dot{y} = x + y\cos(x^2 + y^2).$$

Find all equilibrium points (i.e. stationary points). Sketch the phase portrait and describe it also briefly by word. What can be said about the stability of the equilibrium point(s)?

It might be helpful to use polar coordinates! (4p)

For information concerning the **voluntary oral exam** *and concerning* **routines for returning the marked written exam** *see the newsforum on the course page.*

Good luck!

$\theta(t)$	$\frac{1}{s}$
$\frac{t^n}{n!}$	$\frac{1}{s^{n+1}}$
e^{-at}	$\frac{1}{s+a}$
$\cosh at$	$\frac{s}{s^2 - a^2}$
$\sinh at$	$\frac{a}{s^2 - a^2}$
$\cos bt$	$\frac{s}{s^2 + b^2}$
$\sin bt$	$\frac{b}{s^2 + b^2}$
$\frac{t}{2b}\sin bt$	$\frac{s}{(s^2 + b^2)^2}$
$\frac{1}{2b^3}(\sin bt - bt\cos bt)$	$\frac{1}{(s^2 + b^2)^2}$
$\frac{a}{\sqrt{4\pi t^3}}\, e^{-a^2/4t}$	$e^{-a\sqrt{s}}$

Table 1.2: Standard transform pairs.