MATEMATISKA INSTITUTIONEN	Written exam
STOCKHOLMS UNIVERSITET	MM5026 VT25
Avd. Matematik	Ordinary differential equations
Examiner: Oliver Petersen	May 28, 2025

No calculator, book or notes are allowed. Give complete justifications for your answers!
At least 14 points (including bonus) are needed in order to proceed to the ***voluntary oral exam****.*

1. (4 points) Consider the ODE

$$x'(t) = x(t)^2,$$
$$x(t_0) = c,$$

where $c \in \mathbb{R}$ is any constant.

(a) Solve the ODE.

(b) What is the largest interval containing t_0 on which there is a continuously differentiable solution $x(t)$?

Solution:

(a) When $c = 0$, the solution is given by $x(t) = 0$ for all t. For $c \neq 0$, we use that the ODE is separable. For t near t_0, we expect that $x(t) \neq 0$ since $x(t_0) = c \neq 0$. We therefore may write

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{x(t)}\right) = -\frac{x'(t)}{x(t)^2} = -1.$$

Integrating this between t_0 and t, we see that

$$\frac{1}{x(t)} - \frac{1}{x(t_0)} = \int_{t_0}^{t} (-1)\mathrm{d}s = -(t - t_0).$$

Using the initial condition $x(t_0) = c$, we conclude that

$$\frac{1}{x(t)} = \frac{1}{c} - (t - t_0),$$

or equivalently

$$x(t) = \frac{1}{\frac{1}{c} - (t - t_0)},$$

for $c \neq 0$. We may now check that indeed

$$x'(t) = \frac{1}{\left(\frac{1}{c} - (t - t_0)\right)^2} = x(t)^2,$$
$$x(t_0) = \frac{1}{\frac{1}{c}} = c,$$

so $x(t)$ is a solution (independent of arguing that we can divide by $x(t)$).

(b) We first note that the solutions found in (a), namely either $x(t) = 0$ if $c = 0$ and

$$x(t) = \frac{1}{\frac{1}{c} - (t - t_0)}$$

if $c \neq 0$. The latter exists for all t except where the denominator vanishes, i.e. for all t such that

$$t \neq t_0 + \frac{1}{c}.$$

We claim that the largest interval where x is the unique solution is

$$(-\infty, \infty),$$

if $c = 0$, and

$$\left(-\infty, t_0 + \frac{1}{c}\right)$$

if $c > 0$, and

$$\left(t_0 + \frac{1}{c}, \infty\right)$$

if $c < 0$. We already have checked that the solution exists on these intervals in the respective cases, so only uniqueness remains. Writing the ODE on the form

$$x'(t) = f(t, x(t)),$$

we note that $f(t, x) = x^2$. By the *local* uniqueness theorem in the book (the global uniqueness theorem does not apply, since f is not Lipshitz on an interval of the form $(t_0 - a, t_0 + a)$), the solution x is the unique solution on any interval containing t_0. The solutions cannot be extended continuously over the end points of the interval, since they do not converge there.

2. (4 points) Solve the system of ODE

$$\mathbf{x}'(t) = \begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix} \mathbf{x}(t) + \begin{pmatrix} 1 \\ 2 \end{pmatrix},$$
$$\mathbf{x}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Solution: Writing

$$A = \begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix},$$

the solution to the homogeneous equation is given by

$$x_h(t) = e^{tA}.$$

There are different ways to compute e^{tA}. The general method is to diagonalize A as far as possible, finding the generalized eigenvalues. That would be one possible way, but a quicker way is to note that

$$A^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

for this particular A. We therefore conclude that

$$e^{At} = \sum_{n=0}^{\infty} \frac{(tA)^n}{n!} = \mathrm{id} + tA = \begin{pmatrix} 1 + 2t & -t \\ 4t & 1 - 2t \end{pmatrix}.$$

We now get a general formula for the solution via the integrating factor method:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-tA}\mathbf{x}(t)\right) = e^{-tA}\left(\mathbf{x}'(t) - A\mathbf{x}(t)\right) = e^{-tA}\begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Integrating both sides between $t_0 = 0$ and t gives

$$e^{-tA}\mathbf{x}(t) - \mathbf{x}(0) = \int_0^t e^{-sA}\begin{pmatrix} 1 \\ 2 \end{pmatrix} \mathrm{d}s.$$

Using the initial condition $\mathbf{x}(0) = 0$, we conclude that

$$
\begin{aligned}
\mathbf{x}(t) &= \int_0^t e^{(t-s)A} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \mathrm{d}s \\
&= \int_0^t \begin{pmatrix} 1+2(t-s) & -(t-s) \\ 4(t-s) & 1-2(t-s) \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \mathrm{d}s \\
&= \int_0^t \begin{pmatrix} 1 \\ 2 \end{pmatrix} \mathrm{d}s \\
&= \begin{pmatrix} t \\ 2t \end{pmatrix}.
\end{aligned}
$$

We double-check that this is indeed the solution:

$$
\begin{aligned}
\mathbf{x}'(t) &= \begin{pmatrix} 1 \\ 2 \end{pmatrix} = A\mathbf{x}(t) + \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \\
\mathbf{x}(0) &= \begin{pmatrix} 1 \cdot 0 \\ 2 \cdot 0 \end{pmatrix} = 0,
\end{aligned}
$$

where we have used that

$$
A \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 0.
$$

3. (4 points) Consider the ODE

$$
xy''(x) + \frac{1}{2}y'(x) + y(x) = 0.
$$

(a) Find the general solution to this ODE using the generalized power series method. It is enough to find the correct recursion formula for the coefficients in the generalized power series expansions.

(b) Are all solutions to this equation continuously differentiable at $x = 0$?

Solution:

(a) We work with the generalized power series Ansatz around $x = 0$:

$$
y(x) = \sum_{k=0}^{\infty} a_k x^{k+\mu}.
$$

The other terms in the ODE are (formally) computed to be

$$
\begin{aligned}
\frac{1}{2}y'(x) &= \sum_{k=0}^{\infty} \frac{1}{2} a_k (k+\mu) x^{k+\mu-1}, \\
xy''(x) &= \sum_{k=0}^{\infty} a_k (k+\mu)(k+\mu-1) x^{k+\mu-1},
\end{aligned}
$$

Now shifting the index to $j = k+1$ (and then relabeling back to k) in the expression for $y(x)$ to match the other terms, we see that

$$
y(x) = \sum_{j=1}^{\infty} a_{j-1} x^{j+\mu-1} = \sum_{k=1}^{\infty} a_{k-1} x^{k+\mu-1}.
$$

Inserting these computations into the ODE gives

$$\begin{aligned}
0 &= \sum_{k=0}^{\infty} a_k(k+\mu)(k+\mu-1)x^{k+\mu-1} + \sum_{k=0}^{\infty} \frac{1}{2}a_k(k+\mu)x^{k+\mu-1} + \sum_{k=1}^{\infty} a_{k-1}x^{k+\mu-1} \\
&= \left(a_0\mu(\mu-1) + \frac{1}{2}a_0\mu\right) x^{\mu-1} \\
&\quad + \sum_{k=1}^{\infty} \left(a_k(k+\mu)(k+\mu-1) + \frac{1}{2}a_k(k+\mu) + a_{k-1}\right) x^{k+\mu-1} \\
&= a_0\mu\left(\mu - \frac{1}{2}\right) x^{\mu-1} + \sum_{k=1}^{\infty} \left(a_k(k+\mu)\left(k+\mu-\frac{1}{2}\right) + a_{k-1}\right) x^{k+\mu-1}.
\end{aligned}$$

This gives us the conditions

$$\begin{aligned}
a_0\mu\left(\mu - \frac{1}{2}\right) &= 0, \\
a_k(k+\mu)\left(k+\mu-\frac{1}{2}\right) &= -a_{k-1},
\end{aligned}$$

for all $k \geq 0$. We may assume that $a_0 \neq 0$, for otherwise we either get $a_k = 0$ for all $k \geq 0$ or we get the same solution as if $a_0 \neq 0$ with shifted indices. Since $a_0 \neq 0$, the first equation implies that

$$\mu = 0$$

or

$$\mu = \frac{1}{2}.$$

For any of these two choices of μ,

$$(k+\mu)\left(k+\mu-\frac{1}{2}\right) \neq 0$$

for $k \geq 1$, and we conclude that

$$a_k = -\frac{a_{k-1}}{(k+\mu)\left(k+\mu-\frac{1}{2}\right)}$$

for all $k \geq 1$.

(b) The numbers $\mu_1 = 0$ and $\mu_2 = 1/2$ are called the *roots of the indicial equation* in the general theory. Since they do not differ by an integer, the theory in the book tells us that the general solution to the linear second order ODE can be written as

$$y(x) = Ax^{\mu_1}y_1(x) + Bx^{\mu_2}y_2(x),$$

where y_1 and y_2 are continuously differentiable at $x = 0$ (they are given by the respective power series expansions above without the factor x^{μ}). Since $x^{\mu_2} = x^{\frac{1}{2}}$ is not continuously differentiable at $x = 0$, it follows that $y(x)$ is a solution to the ODE which is not continuously differentiable at $x = 0$ if, for example, $A = 0$ and $B = 1$. **Answer:** Not all solutions to the ODE are continuously differentiable at $x = 0$.

4. (4 points) Solve the following initial value problem using the Laplace transform:

$$\begin{aligned} y''(t) - y(t) &= 3e^{3t}, \\ y(0) &= 2, \\ y'(0) &= 1. \end{aligned}$$

Solution: Let $Y(s) = \mathcal{L}[y(t)](s)$. Applying the Laplace transform to both sides of the equation yields

$$\begin{aligned} \mathcal{L}[y''(t) - y(t)](s) &= \mathcal{L}[3e^{3t}](s) \\ s^2 Y(s) - sy(0) - y'(0) - Y(s) &= \frac{3}{s-3} \\ (s^2-1)Y(s) - 2s - 1 &= \frac{3}{s-3} \\ Y(s) &= \frac{3}{(s-3)(s^2-1)} + \frac{2s}{s^2-1} + \frac{1}{s^2-1} \end{aligned}$$

Using partial fraction decomposition we obtain

$$\frac{3}{(s-3)(s^2-1)} = \frac{3}{8}\frac{1}{s-3} - \frac{3}{8}\frac{s}{s^2-1} - \frac{9}{8}\frac{1}{s^2-1},$$

and hence

$$Y(s) = \frac{3}{8}\frac{1}{s-3} + \frac{13}{8}\frac{s}{s^2-1} - \frac{1}{8}\frac{1}{s^2-1}.$$

Using injectivity of the Laplace transform and the formulas

$$\begin{aligned} \mathcal{L}[\cosh(t)] &= \frac{s}{s^2-a^2} \\ \mathcal{L}[\sinh(at)] &= \frac{a}{s^2-a^2} \\ \mathcal{L}[e^{at}] &= \frac{1}{s-a}, \end{aligned}$$

we obtain the final solution

$$y(t) = \frac{3}{8}e^{3t} + \frac{13}{8}\cosh(t) - \frac{1}{8}\sinh(t).$$

Alternatively, we may obtain the solution without using the Laplace transform formulas for sinh or cosh by taking the partial fraction decomposition of $\frac{1}{s^2-1}$ to obtain

$$Y(s) = \frac{3}{8}\frac{1}{s-3} + \frac{7}{8}\frac{1}{s+1} + \frac{6}{8}\frac{1}{s-1},$$

which gives us

$$y(t) = \frac{3}{8}e^{3t} + \frac{7}{8}e^{-t} + \frac{6}{8}e^{t}.$$

5. (4 points) Let $c_1, c_2 \in \mathbb{R}$ and consider the boundary value problem

$$\begin{aligned} y''(x) + 2y'(x) + 2y(x) &= 0, \\ y(0) &= c_1, \\ y(L) &= c_2. \end{aligned}$$

For what $L > 0$ does there exist a unique solution?

Solution: By the general theory in the book, the boundary value problem has a unique solution if and only if the associated homogeneous boundary value problem has only the trivial solution, i.e. when $c_1 = c_2 = 0$. We proceed by solving the associated homogeneous boundary value problem.

The characteristic polynomial of the given differential equation is $\lambda^2 + 2\lambda + 2$. Rewriting this as $(\lambda + 1)^2 + 1$, we see the roots are $\lambda = -1 \pm i$, and hence the general solution to the differential equation is

$$y(t) = e^{-t}(a\cos(t) + b\sin(t)) \quad a, b \in \mathbb{R}.$$

The first boundary value $y(0) = 0$ is only satisfied if $a = 0$, so we obtain a one parameter family of functions satisfying this condition

$$y(t) = be^{-t}\sin(t) \quad b \in \mathbb{R}.$$

The second boundary value $y(L) = 0$ gives us the equation

$$be^{-L}\sin(L) = 0,$$

which is satisfied when either $b = 0$ or $\sin(L) = 0$. Thus if $\sin(L) \neq 0$, the only solution to the boundary value problem is the trivial solution. Further, if $\sin(L) = 0$, then each function in the one parameter family given above is a solution, and hence the solution is not unique. Therefore, we conclude that the boundary value problem has a unique solution for all $L > 0$ such that $L \neq n\pi$ where $n \in \mathbb{N}$.

6. (4 points) Let E denote the function

$$E(x_1, x_2) := x_1^2 + \sin^2(x_2).$$

(a) Find an autonomous system with an equilibrium point $\hat{x}$, for which E is a strict Liapunov function in some open set $\Omega \subseteq \mathbb{R}^2$ containing $\hat{x}$.
(b) Sketch the orbits of your autonomous system.
(c) Find all equilibrium points of your autonomous system.
(d) Determine if the equilibrium points are unstable, stable and/or asymptotically stable.

Solutions:

(a) We claim that E is a strict Liapunov function in the open set

$$\Omega := \left\{ (x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 < \frac{1}{100} \right\}$$

for the autonomous system

$$\mathbf{x}'(t) = -\mathbf{x}(t)$$

with equilibrium point

$$\hat{x} := \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

It is clear that $\mathbf{x}'(t) = 0$ at $\hat{x}$, so it is an equilibrium point. Moreover, $E(x_1, x_2) \geq 0$ for all $(x_1, x_2) \in \mathbb{R}$ and only vanishes at $x_1 = 0$ and $x_2 = k\pi$, where $k \in \mathbb{Z}$. Note that $\hat{x}$ corresponds to the case when $k = 0$, so $\hat{x} \in \Omega$. However, for $k \neq 0$, we compute that

$$x_1^2 + x_2^2 = 0^2 + (k\pi)^2 = k^2\pi^2 > \pi^2 > \frac{1}{100},$$

so all other zeros of E do not lie in Ω. Finally, we check that $E(x_1(t), x_2(t))$ has negative derivative along any orbit:

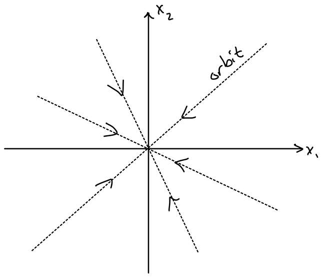
$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E(x_1(t), x_2(t)) &= 2x_1(t)x_1'(t) + 2\sin(x_2(t))\cos(x_2(t))x_2'(t) \\ &= -2x_1(t)^2 - \sin(2x_2(t))x_2(t) \\ &= -2\left(x_1(t)^2 + \frac{\sin(2x_2(t))}{2x_2(t)}x_2(t)^2\right) \\ &< 0\end{aligned}$$

unless $(x_1(t), x_2(t)) = (0, 0)$, since

$$\frac{\sin(2x_2(t))}{2x_2(t)} > \frac{1}{2}$$

for $x_2 \in (0, 1/2)$ is a standard inequality (one can also argue using Taylor's theorem).

(b)



(c) The equilibrium points are given by all $\mathbf{x} \in \mathbb{R}^2$ such that $\mathbf{f}(\mathbf{x}) = 0$, where we write our system as

$$\mathbf{x}'(t) = \mathbf{f}(\mathbf{x}(t)).$$

In our case, $\mathbf{f}(\mathbf{x}) = -\mathbf{x}$, so there is only one equilibrium point given by $\mathbf{x} = \hat{x} = 0$.

(d) Our autonomous system can be written on the form

$$\mathbf{x}'(t) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{x}(t),$$

so it is a linear system. Therefore the linearization at the equilibrium point $\hat{x}$ and it has two negative eigenvalues that coincide, namely -1. The general theory in the course book therefore implies that the equilibrium point is *asymptotically stable.*