MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Avd. Matematik
Examiner: Oliver Petersen

Written exam
MM5026 VT26
Ordinary differential equations
May 26, 2026

No calculator, book or notes are allowed. Give complete justifications for your answers!
At least 14 points (including bonus) are needed in order to proceed to the ***voluntary oral exam****.*

1. (4 points) Consider the initial value problem

$$
\begin{aligned}
x'(t) &= te^{x(t)},\\
x(0) &= 1.
\end{aligned}
$$

   (a) What is the solution?
   (b) For what $t \in \mathbb{R}$ is the solution defined?

   Solution: This is a separable ODE. Since e^x is nowhere zero, we can rewrite the ODE as

$$\frac{\mathrm{d}}{\mathrm{d}t}e^{-x(t)} = -x'(t)e^{-x(t)} = -\frac{x'(t)}{e^{x(t)}} = -t.$$

   Integrating both sides between 0 and t gives

$$e^{-x(t)} - e^{-x(0)} = -\frac{t^2}{2}.$$

   Inserting the initial condition implies that

$$e^{-x(t)} = e^{-1} - \frac{t^2}{2}.$$

   We thus get

$$x(t) = -\ln\left(e^{-1} - \frac{t^2}{2}\right),$$

   which is defined for all $t \in (-\sqrt{2e^{-1}}, \sqrt{2e^{-1}})$.
   (*Check that this is indeed the solution!*)

2. (4 points)

   (a) Show that the Laplace transform of the function $f(t) = t$ is $\mathcal{L}[f](s) = \frac{1}{s^2}$ for all $s > 0$.
   (b) Solve the following initial value problem using the Laplace transform:

$$
\begin{aligned}
y''(t) - y(t) &= t,\\
y(0) &= 0,\\
y'(0) &= -1.
\end{aligned}
$$

   Solution:

   (a) By definition of the Laplace transform,

$$\mathcal{L}[f](s) = \int_0^\infty te^{-st}\mathrm{d}t = -\frac{1}{s}[te^{-st}]_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}\mathrm{d}t = -\frac{1}{s^2}[e^{-st}]_0^\infty = \frac{1}{s^2}.$$

(b) From the formulas in the lecture (or by deriving them again), we compute

$$\begin{aligned}
\mathcal{L}[y'](s) &= s\mathcal{L}[y](s) - y(0) \\
&= s\mathcal{L}[y](s), \\
\mathcal{L}[y''](s) &= s^2\mathcal{L}[y](s) - sy(0) - y'(0) \\
&= s^2\mathcal{L}[y](s) + 1.
\end{aligned}$$

Laplace transforming both sides of the ODE, using problem (a), we get

$$(s^2 - 1)\mathcal{L}[y](s) + 1 = \frac{1}{s^2},$$

which is equivalent to

$$\mathcal{L}[y](s) = \frac{\frac{1}{s^2} - 1}{s^2 - 1} = \frac{1 - s^2}{s^2(s^2 - 1)} = -\frac{1}{s^2}.$$

We conclude that $y(t) = -t$.
(*Check that this is indeed the solution!*)

3. (4 points) Find one solution to the system of ODE

$$\mathbf{x}'(t) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \mathbf{x}(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Solution 1: Let

$$A := \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

and note that $A^2 = A$. This means that

$$e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!} = \mathrm{id} + \sum_{k=1}^{\infty} \frac{A t^k}{k!} = \mathrm{id} + A \sum_{k=1}^{\infty} \frac{t^k}{k!} = \mathrm{id} - A + Ae^t.$$

We get a particular solution by the formula

$$\begin{aligned}
\mathbf{x}_p(t) &= \int_0^t e^{A(t-s)} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mathrm{d}s \\
&= \int_0^t \begin{pmatrix} 0 \\ 1 \end{pmatrix} + (e^{t-s} - 1)A \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mathrm{d}s \\
&= \begin{pmatrix} 0 \\ t \end{pmatrix} + \int_0^t \begin{pmatrix} e^{t-s} - 1 \\ 0 \end{pmatrix} \mathrm{d}s \\
&= \begin{pmatrix} -1 + e^t - t \\ t \end{pmatrix}.
\end{aligned}$$

Solution 2: The second row is equivalent to $x_2'(t) = 1$, so we may choose $x_2(t) = t$. Inserting this in the first row gives

$$x_1'(t) = x_1(t) + x_2(t) = x_1(t) + t.$$

This can be solved by the integrating factor method, which for example gives

$$x_1(t) = e^t - t - 1.$$

4. (4 points) Consider the autonomous system

$$\begin{aligned} x'(t) &= ay(t) + bx(t)y(t), \\ y'(t) &= cx(t)y(t) + dx(t), \end{aligned}$$

for some real numbers a, b, c and d.

(a) Choose real numbers a, b, c, d such that this system has precisely 2 equilibrium points, and compute the location of the equilibrium points.

(b) Is the origin an unstable, stable and/or asymptotically stable equilibrium point for your system?

Solution:

(a) We choose $a = b = c = d = 1$. Then the system becomes

$$\begin{aligned} x'(t) &= y(t) + x(t)y(t) = y(t)(1 + x(t)), \\ y'(t) &= x(t)y(t) + x(t) = x(t)(y(t) + 1), \end{aligned}$$

which has equilibrium points precisely when $x = y = 0$ or $x = y = -1$.

(b) The linearized system at the origin, which indeed is an equilibrium point, is given by

$$\begin{aligned} x'(t) &= y(t), \\ y'(t) &= x(t). \end{aligned}$$

A solution is given by $x(t) = y(t) = e^t$, which converges to the origin as $t \to -\infty$ and goes to infinity along the diagonal in the first quadrant (where $x, y \geq 0$), as $t \to \infty$. Therefore this equilibrium point is *unstable*.

5. (4 points) Consider the ODE

$$x^2 y''(x) - \frac{x}{2} y'(x) + \frac{1}{2} y(x) = 0,$$

for $x > 0$.

(a) Find the general solution to this ODE near $x = 0$ using the generalized power series method.

(b) Do all solutions to this ODE extend to continuously differentiable functions at $x = 0$?

Solution:

(a) The generalized power series method starts with the following Ansatz:

$$y(x) = \sum_{k=0}^{\infty} a_k x^{k+\mu}.$$

Computing that

$$\begin{aligned} y'(x) &= \sum_{k=0}^{\infty} (k + \mu) a_k x^{k+\mu-1}, \\ y''(x) &= \sum_{k=0}^{\infty} (k + \mu)(k + \mu - 1) a_k x^{k+\mu-2}. \end{aligned}$$

Inserting this into the ODE gives

$$\sum_{k=0}^{\infty}\left((k+\mu)(k+\mu-1)-(k+\mu)\frac{1}{2}+\frac{1}{2}\right)a_k x^{k+\mu}=0,$$

which is equivalent to

$$(k+\mu)(k+\mu-1)-(k+\mu)\frac{1}{2}+\frac{1}{2}=0 \tag{1}$$

or $a_k = 0$, for any given k. Without loss of generality (since otherwise we could just change μ), we may assume that $a_0 \neq 0$. This means that

$$\mu(\mu-1)-\frac{\mu}{2}+\frac{1}{2}=0.$$

The roots are given by

$$\mu=\frac{3}{4}\pm\sqrt{\frac{9}{16}-\frac{1}{2}}=\frac{3}{4}\pm\frac{1}{4}=\frac{1}{2}\text{ or }1.$$

In case $\mu = 1$, (1) simplifies to

$$k\left(k+\frac{1}{2}\right)=0,$$

which only has the solution $k = 0$, since k needs to be an integer. Therefore, for $\mu = 0$ and all other choices of k, then the left-hand side of (1) is non-zero. Hence $a_k = 0$ for all $k \neq 0$, when $\mu = 0$. Similarly, in case $\mu = \frac{1}{2}$, (1) simplifies to

$$k\left(k-\frac{1}{2}\right)=0.$$

The only integer solution is again $k = 0$. Hence $a_k = 0$ for all $k \neq 0$, when $\mu = \frac{1}{2}$. The conclusion is that the general solution is

$$y(x)=Ax+B\sqrt{x}.$$

(b) No, if $B \neq 0$, then $y'(x) = A + \frac{1}{2}Bx^{-\frac{1}{2}}$ does not have a (finite) limit when $x \to 0$.

6. (4 points) Let $c_1, c_2 \in \mathbb{R}$ and consider the boundary value problem

$$\begin{aligned} y''(x)+ay'(x)+y(x)&=\sin(x),\\ y(0)&=c_1,\\ y(\pi)&=c_2, \end{aligned}$$

for some real number a. Show that if $|a| > 2$, then there is a unique solution to this problem, defined for all $x \in [0, \pi]$.

Solution: By a theorem in the lectures, it suffices to consider the boundary value problem

$$\begin{aligned} y''(x)+ay'(x)+y(x)&=0,\\ y(0)&=0,\\ y(\pi)&=0. \end{aligned}$$

The general solution to the ODE is given by

$$y(x) = Ae^{r_+ x} + Be^{r_- x},$$

where

$$r_\pm = -\frac{a}{2} \pm \sqrt{\frac{a^2}{4} - 1}.$$

Since $|a| > 2$, both roots are real and distinct. The boundary conditions imply that

$$\begin{aligned} 0 = y(0) = &\ A + B, \\ 0 = y(\pi) = &\ Ae^{r_+ \pi} + Be^{r_- \pi}. \end{aligned}$$

Since $r_- \neq r_+$ and $r_\pm$ are real, it follows that

$$e^{r_+ \pi} \neq e^{r_- \pi}.$$

Therefore, it follows that $A = B = 0$, which means that $y = 0$ is the unique solution.

Good luck!