MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Avd. Matematik
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Written exam
MM5026 VT26
Ordinary differential equations
August 28, 2026

No calculator, book or notes are allowed. Give complete justifications for your answers!
At least 14 points (including bonus) are needed in order to proceed to the ***voluntary oral exam****.*

1. (4 points) Consider the initial value problem

$$x'(t) + x(t)^2 = 1,$$
$$x(0) = \frac{1}{2}.$$

(a) How many solutions does the above initial value problem have?

(b) Determine all solutions!

(c) Specify for each solution the maximal interval of existence!

Solution:

(a) The theorem about existence and uniqueness can be applied, as the function $F(t,x) := 1 - x^2$ satisfies a Lipschitz-condition in a neighbourhood of the point $(t,x) = (0, \frac{1}{2})$.
Answer: The above initial value problem has exactly one solution.

(b) Firstly, let us note that both $x(t) \equiv 1$ and $x(t) \equiv -1$ are solutions, but they do not satisfy our initial condition. By the uniqueness of the solution we may therefore, assume that $x(t) \neq \pm 1$ from now on. We rewrite the ODE as

$$x'(t) = 1 - x(t)^2,$$

which is equivalent to

$$\frac{x'(t)}{1 - x(t)^2} = 1,$$

since $x(t) \neq \pm 1$ for all t in the existence interval. We may rewrite this as

$$\frac{x'(t)}{1 - x(t)} + \frac{x'(t)}{1 + x(t)} = 2,$$

or

$$-\frac{\mathrm{d}}{\mathrm{d}t} \ln |1 - x(t)| + \frac{\mathrm{d}}{\mathrm{d}t} \ln |1 + x(t)| = 2,$$

and integrate both sides to get

$$\ln \left| \frac{1 + x(t)}{1 - x(t)} \right| = 2t + C,$$

which gives

$$\frac{1 + x(t)}{1 - x(t)} = \tilde{C} e^{2t}.$$

The initial condition implies that $\tilde{C} = 3$ and solving for $x(t)$ finally gives

$$x(t) = \frac{\tilde{C} e^{2t} - 1}{\tilde{C} e^{2t} + 1}.$$

Answer: $x(t) = \frac{3e^{2t} - 1}{3e^{2t} + 1}$.

(c) The solution exists for all $t \in \mathbb{R}$, since the denominator is never zero.

2. (4 points) Consider the ODE

$$(x-1)y''(x) + 2xy'(x) - 3y(x) = 0.$$

(a) Do all solutions to this ODE sufficiently near $x = 0$ extend to continuously differentiable functions at $x = 0$?

(b) Find the general solution to this ODE near $x = 0$ using the power series method. It is enough to find the recursion relation.

Solution:

(a) Yes, since $x - 1 \neq 0$ for sufficiently small x, we may rewrite the ODE on standard form

$$y''(x) + \frac{2x}{x-1}y'(x) - \frac{3}{x-1}y(x) = 0$$

near $x = 0$. For $|x| < 1$, the ODE is *linear* and the coefficients are *smooth*, so the existence and uniqueness theory for initial value problems imply that all solutions extend with continuous derivative (in fact smoothly) to $x = 0$.

Answer: Yes.

(b) Note here that $x = 0$ is not a singular point, so we may work with the standard power series Ansatz

$$\begin{aligned} y(x) &= \sum_{k=0}^{\infty} a_k x^k, \\ \Rightarrow\ y'(x) &= \sum_{k=1}^{\infty} k a_k x^{k-1}, \\ \Rightarrow\ y''(x) &= \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2}. \end{aligned}$$

Inserting this into the ODE, we find

$$\sum_{k=2}^{\infty} k(k-1)a_k x^{k-1} - \sum_{k=2}^{\infty} k(k-1)a_k x^{k-2} + 2\sum_{k=1}^{\infty} k a_k x^k - 3\sum_{k=0}^{\infty} a_k x^k = 0.$$

Shifting the indices in the first two sums, we get

$$\sum_{k=1}^{\infty}(k+1)k a_{k+1} x^k - \sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}x^k + 2\sum_{k=1}^{\infty} k a_k x^k - 3\sum_{k=0}^{\infty} a_k x^k = 0,$$

from which we conclude that

$$\begin{aligned} 2a_2 - 3a_0 &= 0, \\ a_{k+2} + \frac{k}{(k+2)}a_{k+1} + \frac{2k-3}{(k+2)(k+1)}a_k, & \end{aligned}$$

for all $k \geq 1$. The recursion relation determines all coefficients $a_2, a_3, \ldots$, given $a_0, a_1 \in \mathbb{R}$.

Answer: The solution is given by $y(x) = \sum_{k=0}^{\infty} a_k x^k$, where $a_0, a_1 \in \mathbb{R}$ are arbitrary and a_k for $k \geq 2$ are given by the recursion relation above.

3. (4 points) Solve the following initial value problem using the Laplace transform:

$$\begin{aligned} y'''(t) + 4y''(t) + 5y'(t) + 2y(t) &= 2e^t, \\ y(0) &= 0, \\ y'(0) &= 0, \\ y''(0) &= 1. \end{aligned}$$

Hint: You may use the algebraic identites

- $s^3 + 4s^2 + 5s + 2 = (s+1)^2(s+2)$, for any $s \in \mathbb{R}$,
- $\frac{1}{(s+1)(s+2)(s-1)} = \frac{1}{6(s-1)} - \frac{1}{2(s+1)} + \frac{1}{3(s+2)}$, for any $s \neq -2, -1, 1$.

Solution:

Using the initial conditions, and assuming suitable decay conditions on y, we compute that

$$\begin{aligned} \mathcal{L}[y'](s) = &\int_0^\infty y'(t)e^{-st}\mathrm{d}t \\ =&[y(t)e^{-st}]_0^\infty + s\int_0^\infty y(t)e^{-st}\mathrm{d}t \\ =&[y(t)e^{-st}]_0^\infty + s\mathcal{L}[y](s), \end{aligned}$$

and similarly $\mathcal{L}[y''](s) = s\mathcal{L}[y](s)$, due to the vanishing initial data. However,

$$\begin{aligned} L[y'''](s) &= \int_0^\infty y'''(t)e^{-st}\mathrm{d}t \\ &= [y''(t)e^{-st}]_0^\infty + s\int_0^\infty y''(t)e^{-st}\mathrm{d}t \\ &= -y''(0) + s\mathcal{L}[y''](s) \\ &= -1 + s^3\mathcal{L}[y](s). \end{aligned}$$

Moreover, we know from the lecture (or compute similarly as above) that $\mathcal{L}[2e^t](s) = \frac{2}{s-1}$. The Laplace transform of entire initial value problem is therefore

$$-1 + \left(s^3 + 4s^2 + 5s + 2\right)\mathcal{L}[y](s) = \frac{2}{s-1},$$

which is equivalent to

$$\left(s^3 + 4s^2 + 5s + 2\right)\mathcal{L}[y](s) = 1 + \frac{2}{s-1} = \frac{s+1}{s-1}.$$

Using the hint, we conclude that

$$\mathcal{L}[y](s) = \frac{1}{(s+1)(s+2)(s-1)} = \frac{1}{6(s-1)} - \frac{1}{2(s+1)} + \frac{1}{3(s+2)}.$$

Since we know that $\mathcal{L}[e^{\alpha t}](s) = \frac{1}{s-\alpha}$, we conclude that the solution is given by

$$y(t) = \frac{1}{6}e^t - \frac{1}{2}e^{-t} + \frac{1}{3}e^{-2t}.$$

(One can easily verify that this is indeed the solution.)

Answer: $y(t) = \frac{1}{6}e^t - \frac{1}{2}e^{-t} + \frac{1}{3}e^{-2t}$.

4. (4 points) Find the general solution to the system of ODEs

$$\mathbf{x}'(t) = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix} \mathbf{x}(t) + \begin{pmatrix} 2 \\ 2 \end{pmatrix}.$$

Solution:

We first note that a particular solution is given by $\mathbf{x}_p(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

To solve the homogeneous problem, let us write

$$A := \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}.$$

One computes that

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}.$$

We decompose the matrix in the middle as the diagonal part

$$D := \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

and the nilpotent part

$$N := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Using that D and N commute, we note that $e^{At} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} e^{Dt} e^{Nt} \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$. It follows that

$$e^{tD} = \sum_{k=0}^{\infty} \frac{D^k t^k}{k!} = \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{2t} \end{pmatrix},$$

and, since $N^2 = 0$,

$$e^{tD} = \sum_{k=0}^{\infty} \frac{N^k t^k}{k!} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}.$$

Multiplying it together, we get

$$e^{At} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \left(e^{2t} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \right) \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1+t & -t \\ t & 1-t \end{pmatrix}.$$

Answer: The general solution is $\mathbf{x}(t) = e^{2t} \begin{pmatrix} 1+t & -t \\ t & 1-t \end{pmatrix} \mathbf{c} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, where $\mathbf{c} \in \mathbb{R}^2$.

5. (4 points) Consider the autonomous system

$$\begin{aligned} x'(t) &= -x(t)y(t)^2 + x(t)^2, \\ y'(t) &= -2y(t) - 1. \end{aligned}$$

For what equilibrium points does there exist a Lyaponov function?

Solution:

The equilibrium points are given by all $(x, y) \in \mathbb{R}^2$ that satisfy

$$x(y^2 - x) = 0, \qquad 2y + 1 = 0,$$

which gives the points $\left(0,-\frac{1}{2}\right)$ and $\left(\frac{1}{4},-\frac{1}{2}\right)$. The linearized system at an equilibrium point (x_0, y_0) is given by

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} -y_0^2 + 2x_0 & -2x_0 y_0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

Inserting the equilibrium point $\left(0,-\frac{1}{2}\right)$, the linearized system is

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} -\frac{1}{4} & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

The eigenvalues are therefore $-\frac{1}{4}$ and -2, which are both negative. Therefore, this equilibrium point is (strictly) stable and there is a (strict) Lyaponov function. For the other equilibrium point $\left(\frac{1}{4},-\frac{1}{2}\right)$, the linearized system is given by

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ 0 & -2 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

Here the eigenvalues are real and have different signs, which means that this is a saddle point, which is unstable. Therefore there is no Lyaponov function in this case.

Answer: There is only a Lyaponov function for the equilibrium point $\left(0,-\frac{1}{2}\right)$.

6. (4 points) Consider the boundary value problem

$$\begin{aligned} y''(x) + 2y'(x) + ay(x) &= 0, \\ y(0) &= 1, \\ y(\pi) &= 2, \end{aligned}$$

for some real number a.

(a) For what $a \in \mathbb{R}$ does there exist a unique solution to this problem, defined for all $x \in [0, \pi]$?

(b) Find all solutions to the boundary value problem for those $a \in (-3, 3)$ for which there does **not** exist a unique solution.

Solution:

(a) By a theorem in the lecture, it suffices to consider the homogeneous problem with trivial data when checking the existence of a unique solution to such a boundary value problem. The characteristic polynomial is given by

$$\lambda^2 + 2\lambda + a = 0,$$

which has the roots

$$\lambda_\pm = -1 \pm \sqrt{1-a}.$$

We therefore have a double root if and only if $a = 1$. In case $a = 1$, the general solution is

$$y(x) = (C_1 + C_2 x)e^{-x}.$$

The first boundary condition implies that

$$0 = y(0) = C_1.$$

The second boundary condition implies that

$$0 = y(\pi) = C_2 \pi e^{-\pi},$$

which implies that $C_2 = 0$, so the solution is unique in this case.
If $a \neq 1$, then the general solution is

$$y(x) = C_1 e^{\lambda_+ x} + C_2 e^{\lambda_- x}.$$

The boundary conditions imply that

$$0 = y(0) = C_1 + C_2, \qquad 0 = y(\pi) = C_1 e^{\lambda_+ \pi} + C_2 e^{\lambda_- \pi},$$

which simplifies to

$$0 = C_1 \left(e^{\lambda_+ \pi} - e^{\lambda_- \pi}\right) = C_1 e^{\lambda_- \pi} \left(e^{(\lambda_+ - \lambda_-)\pi} - 1\right) = C_1 e^{\lambda_- \pi} \left(e^{2\pi\sqrt{1-a}} - 1\right)$$

which either implies that $C_1 = 0$, i.e. the trivial solution, or that $\sqrt{1-a}$ is an non-zero integer times the imaginary number i (as usual in this context, $\sqrt{1-a}$ means $i\sqrt{|1-a|}$ if $1-a$ is negative). This is equivalent to

$$a = 1 + k^2,$$

where $k \in \mathbb{Z}\backslash\{0\}$.
Answer: There exists a unique solution precisely if $a \neq 1 + k^2$, where k is a non-zero integer.

(b) The only relevant case is when $a = 2$. By (a), the general solution is given by

$$y(x) = C_1 e^{(-1+i)x} + C_2 e^{(-1-i)x}.$$

The boundary conditions imply that

$$1 = y(0) = C_1 + C_2,$$

and

$$2 = y(\pi) = C_1 e^{(-1+i)\pi} + C_2 e^{(-1-i)\pi} = e^{(-1+i)\pi} \left(C_1 + C_2\right).$$

We thus conclude that

$$1 = C_1 + C_2 = 2e^{(-1+i)\pi} = -2e^{-1},$$

which is a contradiction.
Answer: There is no solution if $a = 2$.