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Exam / Tentamen
Type Theory
MM8036, 7,5 hp, VT2025
Thursday 2025-05-22

This exam consists of 5 questions, worth a total of 40 points. Note that some parts are intended to be quite challenging; you do not need to complete all questions in order to achieve a good grade. You may submit your answers in either English or Swedish. Write clearly and motivate your answers carefully.

There is no need to give tree-style typings derivations, provided your notation for terms is clear enough to be understood without them.

Good luck! — Lycka till!

1. The *Ackermann function* $A : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is defined as follows:

$$\begin{aligned} A(0, n) &\coloneqq n + 1 \\ A(m + 1, 0) &\coloneqq 1 \\ A(m + 1, n + 1) &\coloneqq A(m, A(m + 1, n)) \end{aligned}$$

   Work in type theory with $\to$, N.

   (a) Define a term $\mathsf{ack} : \mathsf{N} \to \mathsf{N} \to \mathsf{N}$ that (intuitively) represents the Ackerman function.

   (b) Show that ack represents it in the precise sense that for all $m, n \in \mathbb{N}$ we have

$$\mathsf{ack}\ \underline{m}\ \underline{n} \rightsquigarrow^* \underline{A(m, n)}$$

   (where $\underline{a}$ denotes the numeral $\mathsf{S}^a\mathsf{o}$ representing some $a \in \mathbb{N}$).

2. In this question, we will show (part of) the extension of our proof of strong normalisation for simple type theory, to cover binary sum types $A + B$.

   Extend the recursive definition of *hereditarily normalising* (HN) by defining a term $\Gamma \vdash t : A + B$ to be HN if every reduction path from t reaches some term either of the form $\mathsf{inl}(a)$, with $a : A$ HN, or $\mathsf{inr}(b)$, with $b : B$ HN.

   (a) Recall the three criteria of the main lemma about HN: (1) any HN term is terminating; (2) any reduct of an HN term is HN; and (3) if a term t is neutral and all its reducts are HN, then t is HN. Extend the inductive proof of this lemma to cover $A + B$: that is, show these three criteria hold for HN at $A + B$ as defined above, assuming that they hold at A and B.

   (b) Recall the main induction on terms: for any term $\Gamma \vdash a : A$, and any substitution $f : \Delta \to \Gamma$, if each term of f is HN, then $\Delta \vdash f^*a : A$ is HN. Extend this proof to cover the rules for $+$.

3. Recall list types in simple type theory, given by the constructors $\mathsf{nil} : \mathsf{List}(A)$, $\mathsf{cons}(a, l) : \mathsf{List}(A)$, and an appropriate elimination principle.

   Give a type in System F representing $\mathsf{List}(A)$ under a Church encoding, and show how this allows translating the constructors and elimination rule above.

4. For each of the following terms of untyped λ-calculus, determine whether it has a normal form; whether it has an infinite reduction path; and whether it is typable in System F. Justify your answers briefly.

   (a) $\lambda x.\, xx$

   (b) $(\lambda x.\, xx)(\lambda x.\, xx)$

   (c) $(\lambda x.\, y)\ ((\lambda x.\, xx)(\lambda x.\, xx))$

5. Work in dependent type theory with at least Id (with the eliminator J) and N; feel free to assume other standard type-formers if desired.

   (a) Show that functions respect equality: that is, for any types $\Gamma \vdash A, B$ type and a function term $\Gamma \vdash f : A \to B$, give a term

   $$\Gamma \vdash \mathsf{ap}_f : (x, x' {:} A) \to \mathsf{Id}(x, x') \to \mathsf{Id}(fx, fx').$$

   (There are several widespread notations for this construction; ap is used in (among other places) the Homotopy Type Theory book, and stands for "action on paths".)

   (b) Show that the successor function on N is injective: that is, give a term of type

   $$(x, y{:}\mathsf{N}) \to \mathsf{Id}(\mathsf{S}x, \mathsf{S}y) \to \mathsf{Id}(x, y).$$

—— End of exam —— Slut på provet ——