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Exam / Tentamen
Type Theory
MM8036, 7,5 hp, VT2025
Thursday 2025-08-14

This exam consists of 3 questions, worth a total of 40 points. Note that some parts are intended to be quite challenging; you do not need to complete all questions in order to achieve a good grade. You may submit your answers in either English or Swedish. Write clearly and motivate your answers carefully.

There is no need to give tree-style typing derivations, provided your notation for terms is clear enough to be understood without them.

Good luck! — Lycka till!

1. In this question, we consider the proof of termination for simple type theory, as presented in class using the notion of "hereditarily normalising", aka "reducible" (following Girard, Lafont and Taylor 1990).

   (a) Give the definition of HN for terms of a product type $\Gamma \vdash t : A \times B$ (assuming that it has been defined for terms of types A and B).

   (b) Recall the three criteria of the main lemma about HN, proved by induction on types: (1) any HN term is terminating; (2) any reduct of an HN term is HN; and (3) if a term t is neutral and all its reducts are HN, then t is HN. Give the case for $A \times B$ in the inductive proof of this lemma: that is, show these three criteria hold for HN at $A \times B$, assuming that they hold at A and B.

   (c) Recall the main induction on terms: for any term $\Gamma \vdash a : A$, and any substitution $f : \Delta \to \Gamma$, if each term of f is HN, then $\Delta \vdash f^* a : A$ is HN. Give the cases of this induction covering the term-formers associated to product types (i.e. pairing and projections).

2. For this question, we work in System F and untyped λ-calculus. Recall the *stripping* or *type erasure* translation from System F to untyped λ-calculus: if $\Gamma \vdash t : A$ in System F, then $|t|$ is the untyped term given by erasing all type information from t. Say that an untyped term is *typable* if it is the stripping of some term of System F.

   (a) Show that "untyped reduction preserves typing": For any term $\Gamma \vdash t : A$ and (untyped, one-step) reduction $|t| \rightsquigarrow s$, there is some term $\Gamma \vdash t' : A$ such that $\Gamma \vdash t \rightsquigarrow t'$ and $|t'| \equiv s$.

   (b) Show that any typable term is strongly normalising (with respect to untyped reduction).

   (c) Show that not every untyped term is typable.

   (d) Either prove or disprove the following statement (dual to part (a)): For any term $\Gamma \vdash t : A$ and (untyped, one-step) reduction $s \rightsquigarrow |t|$, there is some term $\Gamma \vdash t' : A$ such that $\Gamma \vdash t' \rightsquigarrow t$ and $|t'| \equiv s$.

3. For this question, work first in dependent type theory with N, Π, Id (and any other standard type-constructors you want), and for later parts in simple type theory with N, $\to$, $\times$, 1 (and

again more if you want). *(If you prefer, you can work in dependent type theory throughout; however, you may need to appeal to results which we covered in class only for simple type theory.)*

In each system, assume the function $\mathsf{add} : \mathsf{N} \to \mathsf{N} \to \mathsf{N}$ has been defined as in class by recursion on its second argument, so that $\mathsf{add}(s, \mathsf{o}) \rightsquigarrow \mathsf{o}$ and $\mathsf{add}(s, \mathsf{S}t) \rightsquigarrow \mathsf{S}(\mathsf{add}(s, t))$.

(a) In dependent type theory, prove that add also respects successor in its *first* argument, up to propositional equality. That is, give a closed term of type

$$\Pi_{x,y:\mathsf{N}} \mathsf{Id}_{\mathsf{N}}(\mathsf{add}(\mathsf{S}x, y), \mathsf{S}(\mathsf{add}(x, y))).$$

(b) In simple type theory, show that add does *not* respect successor in its first argument up to judgemental equality (the symmetric-transitive closure of reduction). That is, show it's *not* the case that for all terms $\Gamma \vdash s, t : \mathsf{N}$, we have

$$\Gamma \vdash \mathsf{add}(\mathsf{S}s, t) = \mathsf{S}(\mathsf{add}(s, t)) : \mathsf{N}.$$

(c) Show however that this does hold for numerals: For any actual natural numbers $m, n \in \mathbb{N}$, we have

$$\Gamma \vdash \mathsf{add}(\mathsf{S}\underline{m}, \underline{n}) = \mathsf{S}(\mathsf{add}(\underline{m}, \underline{n})) : \mathsf{N}.$$

—— End of exam —— Slut på provet ——