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Exam / Tentamen
Type Theory
MM8036, 7,5 hp, VT2025
Monday 2026-08-24

This exam consists of 4 questions, worth a total of 40 points. Note that some parts are intended to be quite challenging; you do not need to complete all questions in order to achieve a good grade. You may submit your answers in either English or Swedish. Write clearly and justify your answers carefully.

There is no need to give tree-style typing derivations, provided your notation for terms is clear enough to be understood without them.

As noted in lectures, presentations of type theories vary in many minor details. Feel free to use whatever presentation you prefer; but if you use one differing significantly from those given in lectures, please note this and give details or a reference.

Good luck! — Lycka till!

1. Recall from homework that a relation $\rightsquigarrow$ on a set X has:

   - the *diamond property* if whenever $t \rightsquigarrow t_1$ and $t \rightsquigarrow t_2$, there is t' with $t_1 \rightsquigarrow t'$ and $t_2 \rightsquigarrow t'$;
   - the *weak diamond property* if whenever $t \rightsquigarrow t_1$ and $t \rightsquigarrow t_2$, there is t' with $t_1 \rightsquigarrow^* t'$ and $t_2 \rightsquigarrow^* t'$;
   - the *Church-Rosser property* if $\rightsquigarrow^*$ satisfies the diamond property.

   (Here $\rightsquigarrow^*$ denotes the reflexive, transitive closure of $\rightsquigarrow$, as usual.)

   (a) Show that if a relation satisfies the Church-Rosser property, then it satisfies the weak diamond property.

   (b) Show that if a relation satisfies the diamond property, then it satisfies the Church–Rosser property.

   (c) Show that β-reduction in untyped λ-calculus does *not* have the diamond property. That is, give a term t with one-step reducts $t \rightsquigarrow t_1$, $t \rightsquigarrow t_2$ that do not have any common 1-step reduct $t_1 \rightsquigarrow t'$, $t_2 \rightsquigarrow t'$. *(We know however from class that $\rightsquigarrow$ satisfies Church-Rosser, so there must be some common multi-step reduct t' with $t_1 \rightsquigarrow^* t'$, $t_2 \rightsquigarrow^* t'$.)*

2. In this question, we extend the proof of strong normalisation for simple type theory to cover the type Bool, with constructors $\mathsf{true}, \mathsf{false} : \mathsf{Bool}$ and an eliminator boolrec, given by the following rules:

$$\Gamma \vdash \mathsf{true} : \mathsf{Bool} \qquad \Gamma \vdash \mathsf{false} : \mathsf{Bool} \qquad \frac{\Gamma \vdash s : C \qquad \Gamma \vdash t : C \qquad \Gamma \vdash b : \mathsf{Bool}}{\Gamma \vdash \mathsf{boolrec}_C(s, t; b) : C}$$

   along with reduction rules specifying (under suitable premises):

$$\mathsf{boolrec}_C(s, t; \mathsf{true}) \rightsquigarrow s \qquad \mathsf{boolrec}_C(s, t; \mathsf{false}) \rightsquigarrow t.$$

(a) By analogy with the definitions given in class/homework for other inductive types, define $HN_\Gamma(\mathsf{Bool})$, and decide which of the term-formers true, false, boolrec to designate as neutral forms.

(b) Recall the three criteria of the main lemma about HN: (1) any HN term is terminating; (2) any one-step reduct of an HN term is HN; and (3) if a term t is neutral and all its one-step reducts are HN, then t is HN.

Show these hold for $HN_\Gamma(\mathsf{Bool})$, using your definitions from part (a).

(c) Recall the main induction on terms: for any term $\Gamma \vdash a : A$ and substitution $f : \Delta \to \Gamma$, if each term of f is HN, then $\Delta \vdash f^* a : A$ is HN. Give the cases of this induction for true, false, and boolrec.

3. Give a Church encoding for Bool in System F. That is, give a type Bool in System F, along with terms true, false, and $\mathsf{boolrec}_C(s, t; b)$ for suitable C, s, t, b, satisfying the typing rules from Question 2 above, and similarly satisfying $\mathsf{boolrec}_C(s, t; \mathsf{true}) \rightsquigarrow^* s$, $\mathsf{boolrec}_C(s, t; \mathsf{false}) \rightsquigarrow^* t$.

4. Work in dependent type theory with identity types Id, and eliminator J; feel free to assume other standard type-formers if desired. Recall the rule for J:

$$\frac{\begin{array}{c}\Gamma \vdash A\ \mathsf{type} \qquad \Gamma,\ x, y{:}A, w{:}\mathsf{Id}_A(x,y) \vdash C\ \mathsf{type} \\ \Gamma,\ z{:}A \vdash d : C[z/x, z/y, \mathsf{r}(z)/w] \qquad \Gamma \vdash a, b : A \qquad \Gamma \vdash p : \mathsf{Id}_A(a,b)\end{array}}{\Gamma \vdash \mathsf{J}(x, y, w.C, z.d; a, b, p) : C[a/x, b/y, p/w]}$$

(a) Give a term witnessing that Id is symmetric: for any $\Gamma \vdash A$ type,

$$\Gamma \vdash \mathsf{sym}_A : (x, y{:}A) \to \mathsf{Id}(x, y) \to \mathsf{Id}(y, x).$$

(b) Give a term similarly witnessing that Id is transitive:

$$\Gamma \vdash \mathsf{trans}_A : (x, y, z{:}A) \to \mathsf{Id}(x, y) \to \mathsf{Id}(y, z) \to \mathsf{Id}(x, z).$$

(c) Show that sym_A from part (a) is (propositionally) its own inverse; that is, for $\Gamma \vdash A$ type, give a term of type

$$\Gamma \vdash (x, y{:}A) \to (p{:}\mathsf{Id}_A(x, y)) \to \mathsf{Id}_{\mathsf{Id}_A(x,y)}\left(\mathsf{sym}_A(y, x, \mathsf{sym}_A(x, y, p)),\ p\right).$$

—— End of exam —— Slut på provet ——