MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Avd. Matematik
Examinator: Annemarie Luger (SU),
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Written exam in
Advanced Real Analysis II,
MM8039 (SF2744)
May 26, 2015
9:00-14:00

The exam consists of two parts:

i) Written exam, which consists of 6 problems (4 points each) and gives you grade C at maximum.
ii) Oral exam, which consists of 2 questions (3 points each). A minimum grade Fx is required for eligibility of the oral exam.

Bonus: *Already passed homework and midterm exam can replace questions according to:*
Passed Homework 1 replaces question 1 in the written exam.
Passed Midterm replaces question 2 in the written exam.
Passed Homework 2 replaces question 3 in the written in exam.
Projects: To be presented on June 8, 10:15–15:00. See course webpage.

Credit scale:

A= at least 27,5 points,	*B= at least 23,5 points,*	*C= at least 21 points,*
D= at least 18 points,	*E= at least 15 points*	*Fx= at least 13,5 points.*

Important*: If you intend to take the oral exam, email to luger@math.su.se no later than Sunday 31/6.*

Motivate your solutions carefully!!!

1. Let X and Y be Banach spaces and $T \in \mathcal{B}(X, Y)$ and denote (as in the lecture notes) the dual operator by $T^+ : Y^* \to X^*$. (Note: in [Friedman] for this operator the notation T^* was used).
   (a) Let $X = Y = \ell^p(\mathbb{N})$ for $1 \le p < \infty$ such that $X^* = \ell^q(\mathbb{N})$ with $\frac{1}{p} + \frac{1}{q} = 1$. Consider the right shift $S \in \mathcal{B}(\ell^p(\mathbb{N}))$ given by
   $$Sx := (0, x_1, x_2, \ldots).$$
   Determine the dual operator S^+.
   (b) If $X = Y$ is a Hilbert space there is also the notion of the adjoint operator $T^* \in \mathcal{B}(X)$. Explain the relation between T^* and T^+ in this case. State all definitions that you are using!

2. Let $\mathcal{H}$ be a Hilbert space and $U \in \mathcal{B}(\mathcal{H})$ be a unitary operator, i.e. $U^* = U^{-1}$.
   (a) Show that all eigenvalues of U lie on the unit circle.
   (b) Show that for $\lambda_0 \in \mathbb{C} \setminus \{0\}$ it holds
   $$\lambda_0 \in \sigma_r(U) \qquad \Longrightarrow \qquad \frac{1}{\overline{\lambda_0}} \in \sigma_p(U).$$
   (c) Which conclusion can be drawn from (a) and (b)?

3. Recall Radon-Nykodym derivative, and prove the following standard properties:
   (a) Linearity: $\frac{d(c_1\nu_1 + c_2\nu_2)}{d\mu} = c_1 \frac{d\nu_1}{d\mu} + c_2 \frac{d\nu_2}{d\mu}$, $c_1, c_2 \in \mathbb{R}$.
   (b) Change of measure: If $\nu \ll \mu$ and g is a ν-integrable function then $\int g d\nu = \int g \frac{d\nu}{d\mu} d\mu$.
   (c) Chain Rule: If $\lambda \ll \nu \ll \mu$ then $\frac{d\lambda}{d\mu} = \frac{d\lambda}{d\nu} \frac{d\nu}{d\mu}$.

Please turn!

4. Recall the definition of a signed measure μ (on a measurable space on the real line) to be (singular) continuous if $\mu(\{x\}) = 0$ for all singletons x. Show that μ is a continuous measure if and only if the function $f(x) := \mu([0, x])$ is continuous.

5. (a) Show that any subset of $\mathbb{R}^n$ with lower Minkowski dimension less than n has Lebesgue measure zero. In particular, any subset $E \subset \mathbb{R}^n$ of positive Lebesgue measure must have full Minkowski dimension $dim_M(E) = n$.

   (b) Prove that the Hausdorff dimension of all rational points in $\mathbb{R}^n$, defined as $\mathbb{Q}^n := \{x = (x_1, \cdots, x_n) : \ x_i \in \mathbb{Q}\}$, is zero.

6. (a) Let a function $K \in C\big([a, b] \times [a, b]\big)$ be given and define

$$\big(Bf\big)(x) := \int_a^b K(x, y) f(y)\, dy.$$

   Show that B is a compact operator in $C[a, b]$, i.e. $B \in \mathcal{K}\big(C[a, b]\big)$.

   (b) Let $\mathcal{H}$ be a Hilbert space and $A \in \mathcal{K}(\mathcal{H})$ be a self adjoint, compact operator. Name the spectral properties of A and explain how these are used in the theory of Sturm-Liouville operators. (You do not have to give all proofs but some are needed in order to get full points!)

If you want to know your result write an email to luger@math.su.se. After correction the marked exams can be picked up in studentexpeditionen, house 6, room 204 (SU).

Good luck!