MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Avd. Matematik
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Written exam in
Advanced Real Analysis II,
MM8039 (SF2744)
August 24, 2017
9:00-14:00

30 *points can be obtained from the written exam* $((3+3) \times 4$ *including bonus) and the oral exam* $(3+3)$.

Credit scale:

A= at least 26,5 points, *B= at least 23 points,* *C= at least 20 points,*
D= at least 17,5 points, *E= at least 15 points* *Fx= at least 13,5 points.*

Important*: If you intend to take the oral exam, email to luger@math.su.se no later than Friday 25/8.*

Motivate your solutions carefully!!!

I-1 Let $\mu_k = \frac{1}{k}\sum_{j=1}^{k} \delta_{j/k}$, where $\delta_{j/k}$ is the Dirac measure with support at j/k. Prove that μ_k converges weakly to the Lebesgue measure on $[0,1]$.

i.e.,

$$\lim_k \mu_k(g) = \int_0^1 g(x)dx$$

for all continuous functions g on $[0,1]$.

I-2 State and prove the Radon-Nikodym theorem.

I-3 Define

$$E = \left\{(0,0), (\frac{1}{m}, 0), (0, \frac{1}{n}), (\frac{1}{m}, \frac{1}{n}) : m, n = 1, 2, 3, \cdots\right\}$$

Find the lower and upper Minkowski dimension of this set.

F-1 Let K be a compact metric space and denote by $X := C(K)$ the Banach space of continuous, complex valued functions on K (equipped with the maximum norm) and fix $a \in X$. Define the operator $A : X \to X$ by

$$(Af)(t) := a(t)f(t) \text{ for } t \in K.$$

(a) Determine $\sigma(A)$.

(b) Give a sufficient condition on a such that $\sigma_p(A) \neq \emptyset$.

(c) Give an example of K and a for which $\sigma_p(A) = \emptyset$.

F-2 Let $\mathcal{H}$ be a Hilbert space and $B \in \mathcal{B}(\mathcal{H})$ be a bounded linear operator.

(a) Show that the set $\sigma(B)$ is compact.

(b) Show that $B = B^*$ is a projection if and only if $\sigma(B) \subset \{0, 1\}$.

Hint: Theorems from the lecture can be used without proof, but it need to clear how they are used!

Please turn!

F-3 Let $\mathcal{H}$ be a Hilbert space. An operator $T \in \mathcal{B}(\mathcal{H})$ is called *normal* if $TT^* = T^*T$.

(a) Show: A linear operator T is normal if and only if $\|Tx\| = \|T^*x\|$ for all $x \in \mathcal{H}$.

(b) Show that for normal operators the following hold:

i. $\ker(T) = \ker(T^*)$

ii. $\operatorname{ran}(T)$ is dense if and only if T is injective.

iii. If $Tx = \mu x$ for some $x \in \mathcal{H}$ and $\mu \in \mathbb{C}$, then $T^*x = \overline{\mu}x$.

iv. If μ and λ are distinct eigenvalues of T, then the corresponding eigenvectors are orthogonal.

After correction the marked exams can be picked up in studentexpeditionen, house 6 (SU).

Good luck!