MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Avd. Matematik
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Exam in
Advanced Real Analysis II,
August 21, 2023

The exam consists of 6 problems, where each problem can give at most 4 points. Grades: 14 points guarantees the grade E, 17 points guarantees the grade D, 20 points guarantees the grade C, 24 points guarantees the grade B. 13 points guarantees the grade Fx. If one gets at least 14 points there is an optional oral exam for higher grades. Information about the voluntary oral exam will be sent out after the exam.
Bonus: Your bonus points are added to Part A.
Maximum score from Part A is 16 points, bonus included.
Aids: No aids allowed.

Part A

1. (a) Let M be a metric space with metric ρ and assume that μ^* is an outer measure on M such that every open set is μ^*-measurable. Show that μ^* is a metric outer measure. (Hint: if the distance between two sets E and F is positive, one can find open sets $E' \supset E$ and $F' \supset F$ such that the distance between E' and F' is also positiv; this fact may be used without proof.)

   (b) Let $\mathcal{A}$ be a σ-algebra (of subsets of X, where X is some non-empty set) and assume that μ_1, μ_2 are two finite measures defined on $\mathcal{A}$. Let $\nu = \mu_1 - \mu_2$ and show that $|\nu|(E) \leq \mu_1(E) + \mu_2(E)$ for every $E \in \mathcal{A}$.

2. Let X be a Banach space.

   (a) Give the definition of what it means that a sequence $(f_n)_{n\in\mathbb{N}}$ in the dual X^* is *weak-$*$ convergent* to $f \in X^*$., i.e. $f_n \overset{*}{\rightharpoonup} f$.

   (b) Show that the weak-$*$ limit of a sequence $(f_n)_{n\in\mathbb{N}} \in X^*$ is unique.

   (c) Given $(f_n)_{n\in\mathbb{N}} \in X^*$ and $f \in X^*$ show
   $$f_n \to f \quad \Longrightarrow \quad f_n \overset{*}{\rightharpoonup} f,$$
   where $f_n \to f$ denotes the norm convergence in X^*.

   (d) Is the converse of (c) true as well?

3. (a) Let $(X, \mathcal{A}, \mu)$ be a finite measure space. Assume that $E_1, \ldots, E_n$ are sets in $\mathcal{A}$ and that $c_1, \ldots, c_n$ are real numbers. For $E \in \mathcal{A}$ we define
   $$\nu(E) = \sum_{k=1}^{n} c_k \mu(E \cap E_k).$$
   Show that ν is a signed measure (on $\mathcal{A}$) such that $\nu \ll \mu$ and find $\dfrac{d\nu}{d\mu}$.

   (b) Let λ, μ, ν be σ-finite measures (on some σ-algebra $\mathcal{A}$ of subsets of X) such that $\nu \ll \mu$ and $\mu \ll \lambda$. Show that
   $$\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu}\frac{d\mu}{d\lambda} \text{ a.e. with respect to } \lambda.$$

4. Let $\mathcal{H}$ be a Hilbert space and $T \in \mathcal{B}(\mathcal{H})$ a bounded linear operator.

   (a) Give the definition of the spectrum of T, i.e. $\sigma(T)$.

   (b) Assume that T is boundedly invertible, i.e. the inverse T^{-1} exists and is bounded. Show that
   $$\sigma(T^{-1}) = (\sigma(T))^{-1} = \left\{ \frac{1}{\lambda} : \lambda \in \sigma(T) \right\}.$$

   (c) Can 0 be an accumulation point of $\sigma(T)$ if T is boundedly invertible? Motivate your answer!

PLEASE TURN!

Part B

5. Assume that μ and ν are two finite measures (on some σ-algebra $\mathcal{A}$ of subsets of X).

   (a) If $\nu \ll \mu$, show that for every $\varepsilon > 0$ there is a $\delta > 0$ such that $\nu(E) < \varepsilon$ for any $E \in \mathcal{A}$ such that $\mu(E) < \delta$. (Hint: what happens if the assertion is false?; the fact that $\nu(\overline{\lim}_{n\to\infty} E_n) \geq \overline{\lim}_{n\to\infty} \nu(E_n)$ for any sequence of sets $E_n \in \mathcal{A}$ may be used without proof.)

   (b) Show that there exist finite measures ν_0 and ν_1 such that $\nu = \nu_0 + \nu_1$, $\nu_0 \perp \mu$ and $\nu_1 \ll \mu$. (Hint: the facts that $\nu \ll \mu + \nu$ and $\int f\, d(\mu+\nu) = \int f\, d\mu + \int f\, d\nu$ may be used without proof.)

6. Assume that $T\colon X \to X$ is a measure preserving transformation on the probability space $(X, \mathcal{A}, \mu)$.

   (a) If T is ergodic w.r.t. to μ, and if ν is a probability measure defined on $\mathcal{A}$ such that T preserves ν and is also ergodic w.r.t. to ν, show that either $\mu = \nu$ or $\mu \perp \nu$.

   (b) If T is mixing, show that T is ergodic.

Good luck!