Räkneövning 15-03-30

6.36 $\iint_{\mathbb{R}^2} \frac{1}{(1+x^2+y^2)^\alpha}\,dxdy$ för vilka $\alpha \in \mathbb{R}$ konvergent? värde?

polära koordinater: $\int_0^\infty\int_0^{2\pi} \frac{r}{(1+r^2)^\alpha}\,dt\,dr = 2\pi \int_0^\infty \frac{r}{(1+r^2)^\alpha}\,dr$

frågan är alltså: för vilka α konvergerar $\int_0^\infty \frac{r}{(1+r^2)^\alpha}\,dr$

$s = r^2+1$ $\quad 0 \rightsquigarrow 1$

$\frac{ds}{dr} = 2r$ $\quad \infty \rightsquigarrow \infty$ $\Rightarrow \int_1^\infty \frac{r}{s^\alpha}\frac{1}{2r}\,ds = \frac{1}{2}\int_1^\infty \frac{1}{s^\alpha}\,ds$

vi vet från Analys A: $\int_1^\infty \frac{1}{s^\alpha}\,ds < \infty \iff \alpha > 1$

och i så fall har vi $\int_1^\infty \frac{1}{s^\alpha}\,ds = \left[\frac{1}{1-\alpha}\frac{1}{s^{\alpha-1}}\right]_1^\infty = \frac{-1}{1-\alpha} = \frac{1}{\alpha-1}$

[formellt: $\int_1^\infty \frac{1}{s^\alpha}\,ds = \lim_{R\to\infty}\int_1^R \frac{1}{s^\alpha}\,ds = \lim_{R\to\infty}\left[\frac{1}{1-\alpha}\frac{1}{s^{\alpha-1}}\right]_1^R$

$= \frac{1}{1-\alpha}\left(\lim_{R\to\infty}\frac{1}{R^{\alpha-1}}\right) - \frac{1}{1-\alpha}\frac{1}{1^{\alpha-1}} = \frac{1}{\alpha-1}$]

Svar: integralen är konvergent omm $\alpha > 1$

i så fall är integralens värde $\frac{\pi}{\alpha-1}$

6.43 $\iint_{\mathbb{R}^2} \frac{x^2}{(1+x^2)(x^2+y^2)^{3/2}}\,dx\,dy$ konvergent eller divergent!?

$$= \int_{-\infty}^{\infty} \frac{\cancel{x}}{(1+x^2)} \int_{-\infty}^{\infty} \frac{x^2}{(x^2+y^2)^{3/2}}\,dy\,dx \overset{*}{=}$$

$x \neq 0$: $\int_{-\infty}^{\infty} \frac{\cancel{x}^2}{(x^2+y^2)^{3/2}}\,dy = \left[\begin{array}{ll} \cos^2 t + \sin^2 t = 1 & \left| \cdot \frac{x^2}{\cos^2 t}\right. \\ x^2 + x^2\tan^2 t = \frac{x^2}{\cos^2 t} & \\ y = x\tan t & \frac{dy}{dt} = x\frac{1}{\cos^2 t} \end{array}\right.$

$-\infty \rightsquigarrow -\frac{\pi}{2}$

$\infty \rightsquigarrow \frac{\pi}{2}$

samma om $x<0$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cancel{x}^2}{(x^2+x^2\tan^2 t)^{3/2}}\, x\frac{1}{\cos^2 t}\,dt = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cancel{x}^2}{\frac{x^3}{\cos^3 t}}\, x\frac{1}{\cos^2 t}\,dt =$$

$$= \frac{\cancel{x}^2}{x^2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos t\,dt = \left[\sin t\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \left[\sin\left(\arctan\frac{y}{x}\right)\right]_{-\infty}^{\infty}$$

$$= \left(\sin\left(\frac{\pi}{2}\right) - \sin\left(-\frac{\pi}{2}\right)\right) = 2$$

$$\overset{*}{=} \int_{-\infty}^{\infty} \frac{\cancel{x}}{1+x^2}\cdot 2\,dx = \int_{-\infty}^{\infty}\frac{2}{1+x^2}\,dx = 4\int_0^{\infty}\frac{1}{1+x^2}\,dx =$$

$$= 4\left[\arctan(x)\right]_0^{\infty} = 4\cdot\frac{\pi}{2} = 2\pi$$

% $x=0$: $\int_{-\infty}^{\infty}\frac{0}{(0+y^2)^{3/2}}\,dy = \int_{-\infty}^{\infty} 0\,dy = 0$

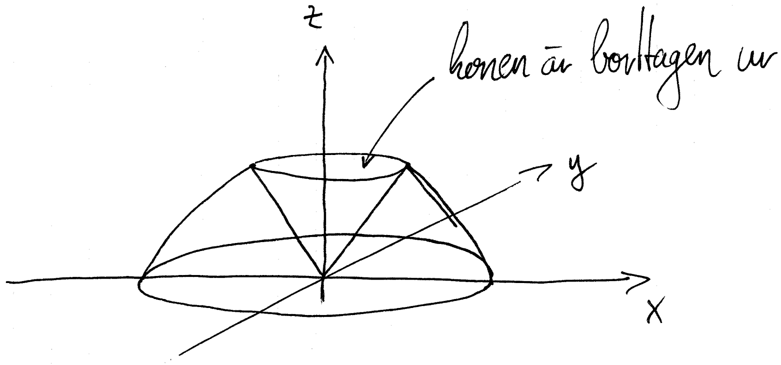
Lättare: $\iint_{\mathbb{R}^2}\frac{x^2}{(1+x^2)(x^2+y^2)^{3/2}}\,dx\,dy = \int_0^{\infty}\int_0^{2\pi}\frac{r^2\cos^2 t}{(1+r^2\cos^2 t)r^3}\,r\,dr\,dt = \int_0^{\infty}\int_0^{2\pi}\frac{\cos^2 t}{1+r^2\cos^2 t}\,dt$

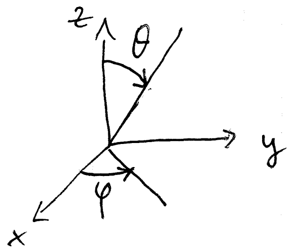
för stort r: $\frac{\cos^2 t}{1+r^2\cos^2 t} \le \frac{1}{r^2} \Rightarrow \int_1^{\infty}\int_0^{2\pi}\frac{\cos^2 t}{1+r^2\cos^2 t}\,dt\,dr \le \int_1^{\infty}\int_0^{2\pi}\frac{1}{r^2}\,dt\,dr < \infty$

för litet r: $\frac{\cos^2 t}{1+r^2\cos^2 t} \le \frac{1}{1+r^2\cos^2 t} \le \frac{1}{1+0} = 1 \Rightarrow \int_0^1\int_0^{2\pi}\frac{\cos^2 t}{1+r^2\cos^2 t}\,dt\,dr \le \int_0^1\int_0^{2\pi} 1\,dt\,dr < \infty$

① $\iiint_D x^2y^2z \, dx\,dy\,dz$

$$D = \{ (x,y,z) \in \mathbb{R}^3 : \ 0 \le z \le \sqrt{x^2+y^2}, \ x^2+y^2+z^2 \le 1 \}$$





$r \in [0,1], \ \varphi \in [0, 2\pi], \ \theta \in [\frac{\pi}{4}, \frac{\pi}{2}]$ rymdpolära koordinat

$$\begin{aligned} x &= r \sin\theta \cos\varphi \\ y &= r \sin\theta \sin\varphi \\ z &= r\cos\theta \end{aligned} \quad \Rightarrow \quad \frac{d(x,y,z)}{d(r,\theta,\varphi)} = r^2 \sin\theta$$

$$\iiint_D x^2y^2z^2 \, dx\,dy\,dz = \int_0^1 \int_0^{2\pi} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} r^2\sin^2\theta\cos^2\varphi \, r^2\sin^2\theta\sin^2\varphi \, r\cos\theta \, r^2 \sin\theta \, d\theta\,d\varphi\,dr$$

$$= \int_0^1 r^7 \, dr \cdot \int_0^{2\pi} \underbrace{\cos^2\varphi \sin^2\varphi}_{=(\cos\varphi\sin\varphi)^2} d\varphi \cdot \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \underbrace{\sin^5\theta\cos\theta \, d\theta}_{=\left(\frac{\sin^6\theta}{6}\right)'}$$

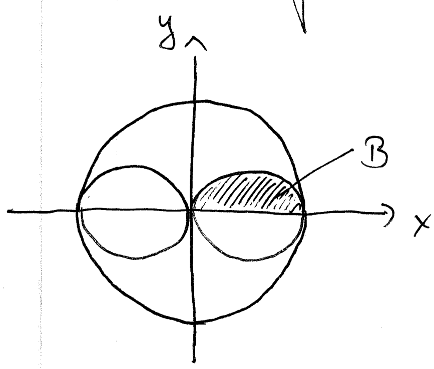
$$= \left(\tfrac{1}{2}\sin(2\varphi)\right)^2 = \tfrac{1}{4}\sin^2(2\varphi) = \tfrac{1}{8}\left(1 - \cos(4\varphi)\right)$$

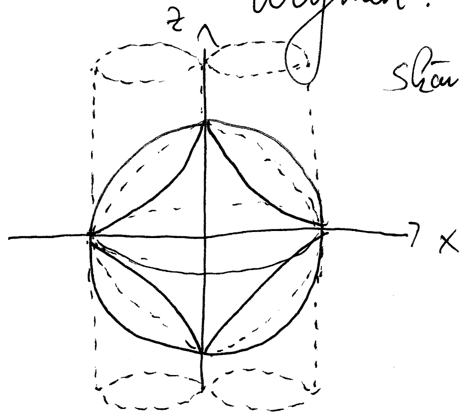
$$= \left[\frac{r^8}{8}\right]_0^1 \cdot \frac{1}{8}\left[\varphi - \frac{1}{4}\sin(4\varphi)\right]_0^{2\pi} \cdot \left[\frac{\sin^6\theta}{6}\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= \frac{1}{8} \cdot \frac{1}{8}(2\pi - 0) \cdot \left(\frac{1 - \left(\frac{1}{\sqrt{2}}\right)^6}{6}\right) = \frac{7\pi}{1536}$$

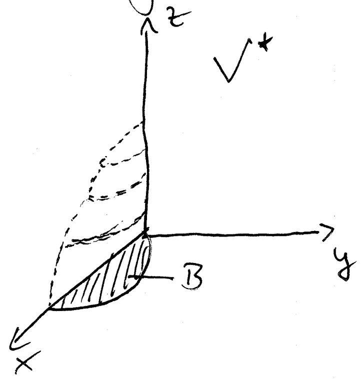
② $D = \{(x,y,z)\in\mathbb{R}^3 : x^2+y^2+z^2 \le 4,\ (x+1)^2+y^2 \ge 1,\ (x-1)^2+y^2 \ge 1\}$

"Vivianis fönster" volymen?





skär bort två cylinder ur ett klot



Låt V^* vara denna delen av klotet som ligger över mängden B, där $B = \{(x,y)\in\mathbb{R}^2 : (x-1)^2+y^2 \le 1\}$

⇝ V^* är en åttondedel av skärningsmängden mellan klotet och cylindrarna

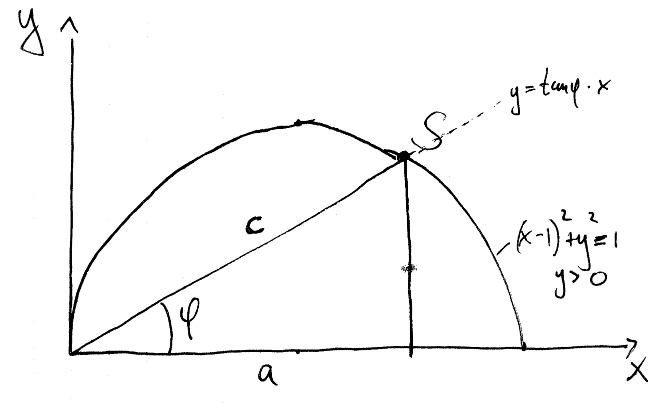
$$\Rightarrow \text{vol}(D) = \text{vol}(\text{Klot}) - 8\,\text{vol}(V^*)$$

$$\text{vol}(\text{Klot}) = \frac{4}{3}2^3\pi = \frac{256}{3}\pi$$

$$\text{vol}(V^*) = \iint_B \left(\int_0^{\sqrt{4-x^2-y^2}} 1\, dz\right) dx\,dy = \iint_B \sqrt{4-x^2-y^2}\, dx\,dy$$

vi vill nu beskriva B i polära koordinater

<u>obs!</u> radien kommer bero på vinkeln ⇝ $\varphi,\ r(\varphi)$



φ är givet ⇝ beräkna $c = r(\varphi)$

$$\cos\varphi = \frac{a}{c}$$

$$c = \frac{a}{\cos\varphi}$$

vad är a?

Skärningspunkten S: $\begin{cases} y = \tan\varphi\, x \\ y = \sqrt{1-(x-1)^2} = \sqrt{1-x^2+2x-1} = \sqrt{2x-x^2} \end{cases}$

$$\Rightarrow \tan\varphi\, x = \sqrt{2x-x^2}$$
$$\tan^2\varphi\, x^2 = 2x - x^2$$
$$x\left(x\left(\tan^2\varphi + 1\right) - 2\right) = 0$$

ointressant!

$$x\left(\frac{\sin^2\varphi + \cos^2\varphi}{\cos^2\varphi}\right) = 2$$
$$x = 2\cos^2\varphi \mathrel{\hat{=}} a$$

$$\Rightarrow c = \frac{2\cos^2\varphi}{\cos\varphi} = 2\cos\varphi$$

$$\Rightarrow \varphi \in \left[0, \frac{\pi}{2}\right] \qquad x = r(\varphi)\cos\varphi$$
$$r \in [0, 2\cos\varphi] \qquad y = r(\varphi)\sin\varphi$$

$$\iint_B \sqrt{4-x^2-y^2}\, dx\, dy = \int_0^{\frac{\pi}{2}} \int_0^{2\cos\varphi} \sqrt{4-r^2}\, r\, dr\, d\varphi = \left[\begin{matrix} 4-r^2 = t & 0 \rightsquigarrow 4 \\ -2r = \frac{dt}{dr} & 2\cos\varphi \rightsquigarrow 4(1-\cos^2\varphi) \end{matrix}\right]$$

$$= \int_0^{\frac{\pi}{2}} \int_4^{4\sin^2\varphi} \sqrt{t}\left(-\frac{1}{2}\right) dt\, d\varphi = \int_0^{\frac{\pi}{2}} \left[\sqrt{t^3}\right]_4^{4\sin^2\varphi} \frac{2}{3}\left(-\frac{1}{2}\right) d\varphi$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{3}\left(8 - 8\sin^3\varphi\right) d\varphi =$$

$$= \frac{8}{3}\int_0^{\frac{\pi}{2}} 1 - \sin^3\varphi\, d\varphi = \frac{8}{3}\left[\varphi + \frac{3\cos\varphi}{4} - \frac{1}{12}\cos(3\varphi)\right]_0^{\frac{\pi}{2}}$$

$$= \frac{8}{3}\left[\frac{\pi}{2} + 0 - 0 - 0 - \frac{3}{4} + \frac{1}{12}\right] = \frac{4\pi}{3} - \frac{16}{9}$$

$$\Rightarrow \operatorname{vol}(D) = \underbrace{\frac{256}{3}\pi - 8\frac{4\pi}{3}}_{=0} + 8\frac{16}{9} = \frac{128}{9}$$