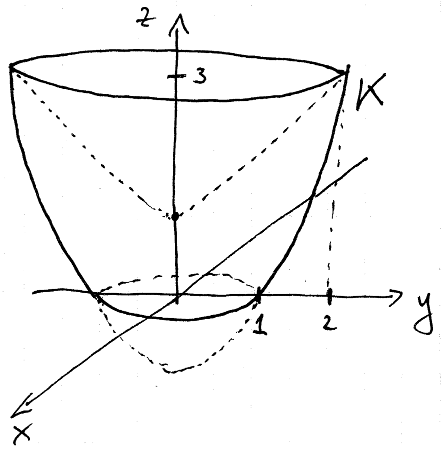
Räkneövning 15-04-01

8.11 volymen av kroppen K som uppfyller

$$0 \leq \underbrace{x^2+y^2-1}_{\to \text{ elliptisk paraboloid}} \leq z \leq \underbrace{\sqrt{x^2+y^2}+1}_{\text{kon}}$$



när gäller $x^2+y^2-1 = \sqrt{x^2+y^2}+1$

$r^2 = x^2+y^2$ $\quad r^2 - 1 = r + 1$

$r^2 - r - 2 = 0$

$r = \frac{1}{2} + \sqrt{\frac{1}{4}+2} = 2$

$\Rightarrow$ höjden av kroppen K är 3

$$vol(K) = \iiint_K 1 \, dx\,dy\,dz = \iint_{x^2+y^2\leq 1} \int_0^{\sqrt{x^2+y^2}+1} 1 \, dz \, dx\,dy + \iint_{1\leq x^2+y^2\leq 4} \int_{x^2+y^2-1}^{\sqrt{x^2+y^2}+1} 1 \, dz \, dx\,dy$$

$$= \iint_{x^2+y^2\leq 1} \sqrt{x^2+y^2}+1 \, dx\,dy + \iint_{1\leq x^2+y^2\leq 4} \sqrt{x^2+y^2} - x^2 - y^2 + 2 \, dx\,dy$$

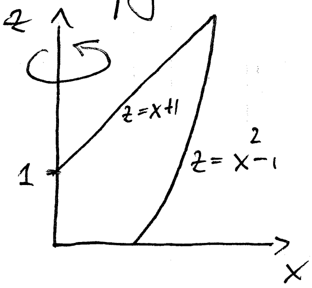
$$= \int_0^1 \int_0^{2\pi} (r+1) r \, dt \, dr + \int_1^2 \int_0^{2\pi} (r - r^2 + 2) r \, dt \, dr$$

$$= 2\pi \left(\left[\frac{r^3}{3} + \frac{r^2}{2} \right]_0^1 + \left[\frac{r^3}{3} - \frac{r^4}{4} + r^2 \right]_1^2 \right) =$$

$$= 2\pi \left(\cancel{\frac{1}{3}} + \frac{1}{2} + \frac{8}{3} - \cancel{\frac{16}{4}} + \cancel{4} - \cancel{\frac{1}{3}} + \frac{1}{4} - 1 \right) =$$

$$= \frac{\pi}{6} \left(6 + 32 + 3 - 12 \right) = \frac{29\pi}{6}$$

Lättare pga rotationssymmetri runt z-axeln:



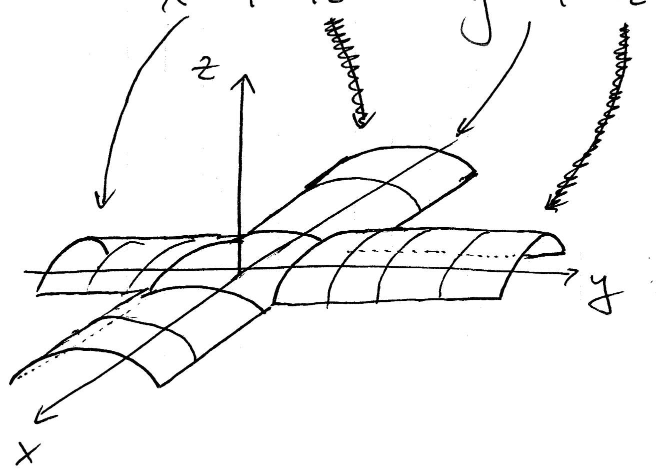
$z = x+1 \iff x = z-1$

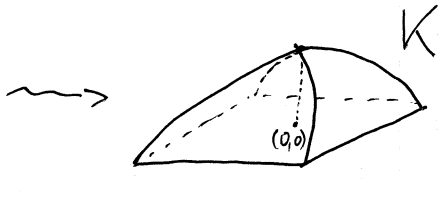
$z = x^2 - 1 \iff x = \sqrt{z+1}$

$\mathrm{vol}(K) = \pi \int f(z)^2 \, dz$

$$\mathrm{vol}(K) = \pi \left(\int_0^1 \left(\sqrt{z+1}\right)^2 dz + \int_1^3 \left(\sqrt{z+1}\right)^2 - (z-1)^2 \, dz \right) = \ldots = \frac{29\pi}{6}$$

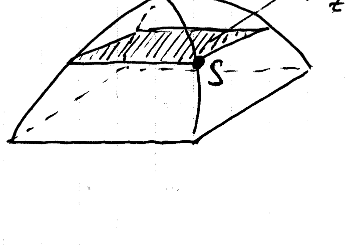
8.8 volymen av kroppen K som begränsas av (paraboliska) cylindrarna $x^2 = 4-4z$ $y^2 = 4-4z$ och xy-planet





höjden: $x = y = 0 \Rightarrow z = 1$

idé: beräkna för varje $z \in [0,1]$ arean $A(z)$ av figuren A_z som uppstår när man skär K med ett plan parallell till xy-planet i höjden z



figuren är en kvadrat

koordinater av S: $\left(\sqrt{4-4z}, \sqrt{4-4z}\right)$

$\Rightarrow$ längden av sidor: $2\sqrt{4-4z}$

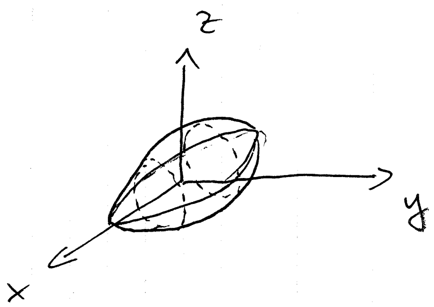
$\Rightarrow A(z) = 4(4-4z) = 16(1-z)$

$$\mathrm{vol}(K) = \int_0^1 \iint_{A_z} 1 \, dx\,dy \, dz = \int_0^1 A(z)\, dz = 16 \int_0^1 1 - z \, dz =$$

$$= 16\left(1 - \frac{1}{2}\right) = 8$$

① $\iiint_D x^2\,dx\,dy\,dz$ D är området $x^2+2y^2+z^2 \leq 2$

D är en ellipsoid



inför "elliptiska rymdpolära" koordinater

$$x = r\sin\theta\cos\varphi \qquad r\in[0,\sqrt{2}]$$
$$y = \frac{1}{\sqrt{2}} r\sin\theta\,\cancel{\sin}\,\varphi \qquad \theta\in[0,\pi]$$
$$z = r\cos\theta \qquad \varphi\in[0,2\pi]$$

$$x^2+2y^2+z^2 = r^2\sin^2\theta\cos^2\varphi + \frac{2}{2}r^2\sin^2\theta\sin^2\varphi + r^2\cos^2\theta = r^2 \leq 2$$

$$\frac{d(x,y,z)}{d(r,\theta,\varphi)} = \begin{vmatrix} \sin\theta\cos\varphi & +r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \frac{1}{\sqrt{2}}\sin\theta\sin\varphi & \frac{r}{\sqrt{2}}\cos\theta\sin\varphi & \frac{r}{\sqrt{2}}\sin\theta\cos\varphi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix} = \frac{1}{\sqrt{2}}\cdot$$ funktional-determinant för vanliga rymdpolära koordinater

$$= \frac{1}{\sqrt{2}} r^2\sin\theta$$

$$\iiint_D x^2\,dx\,dy\,dz = \int_0^{\sqrt{2}}\int_0^{2\pi}\int_0^{\pi} r^2\sin^2\theta\cos^2\varphi\,\frac{1}{\sqrt{2}}r^2\sin\theta\,d\theta\,d\varphi\,dr =$$

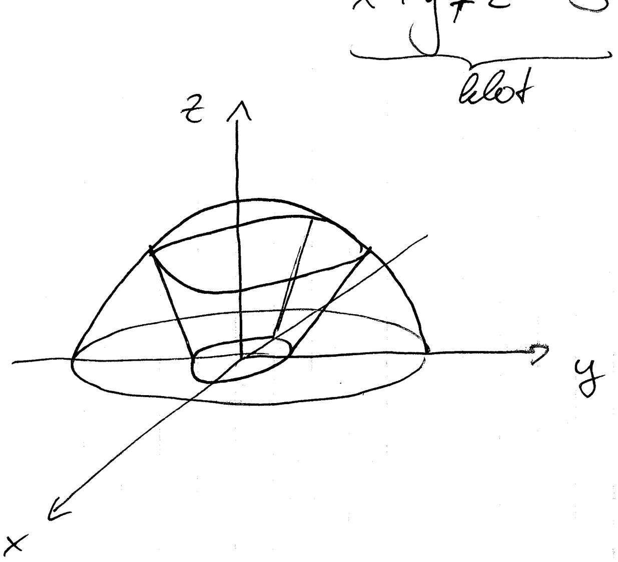
$$= \frac{1}{\sqrt{2}}\int_0^{\sqrt{2}} r^4\,dr \int_0^{2\pi}\cos^2\varphi\,d\varphi \int_0^{\pi}\sin^3\theta\,d\theta =$$

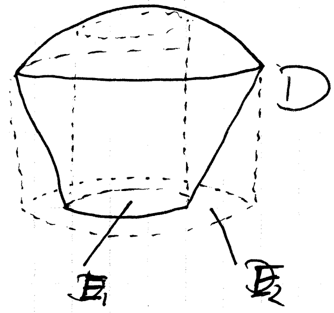
$$= \frac{1}{\sqrt{2}}\left[\frac{r^5}{5}\right]_0^{\sqrt{2}} \left[\frac{\varphi+\sin\varphi\cos\varphi}{2}\right]_0^{2\pi} \left[\frac{\cos(3\theta)-9\cos\theta}{12}\right]_0^{\pi}$$

$$= \frac{1}{\sqrt{2}}\;\frac{4\sqrt{2}}{5}\;\;\pi\;\;\frac{4}{3} = \frac{16\pi}{15}$$

② $\iiint_D z \, dx\,dy\,dz$ D är området $z \geq 0$, $\underbrace{z^2 \geq 2x^2+3y^2-1}_{\text{"elliptisk" kon}}$

$\underbrace{x^2+y^2+z^2 \leq 3}_{\text{klot}}$





beskriva E_1 och E_2 $\subseteq$ xy-planet

rand av E_1: $0 = 2x^2+3y^2-1$

$\Rightarrow E_1 = \{(x,y) : 2x^2+3y^2 \leq 1\}$

rand av E_2: $\begin{cases} z^2 = 2x^2+3y^2-1 \\ x^2+y^2+z^2 = 3 \end{cases}$

$\Rightarrow 3x^2+4y^2 = 4$

$\Rightarrow E_2 = \{(x,y) : 3x^2+4y^2 \leq 4\} \setminus E_1$

$\Rightarrow$ dela upp $D = D_1 \cup D_2$

$D_1 = \{(x,y,z) : (x,y) \in E_1,\ 0 \leq z \leq \sqrt{3-x^2-y^2}\}$

$D_2 = \{(x,y,z) : (x,y) \in E_2,\ \sqrt{2x^2+3y^2-1} \leq z \leq \sqrt{3-x^2-y^2}\}$

$$\iiint_D z\,dx\,dy\,dz = \iiint_{D_1} z\,dx\,dy\,dz + \iiint_{D_2} z\,dx\,dy\,dz$$

$$= \iint_{E_1}\left(\int_0^{\sqrt{3-x^2-y^2}} z\,dz\right)dx\,dy + \iint_{E_2}\left(\int_{\sqrt{2x^2+3y^2-1}}^{\sqrt{3-x^2-y^2}} z\,dz\right)dx\,dy$$

$$= \frac{1}{2}\iint_{E_1} 3-x^2-y^2\,dx\,dy + \frac{1}{2}\iint_{E_2} 3-x^2-y^2-2x^2-3y^2+1\,dx\,dy$$

$$= \frac{1}{2}\left(\iint_{E_1} 3-x^2-y^2\,dx\,dy + \iint_{E_2} 4-3x^2-4y^2\,dx\,dy\right)$$

isället för att parametrisera E_2 direkt (obs: randkurvorna är inte helt lika) så räknar vi integralen över $E_2 \cup E_1$ minus den över E_1 i andra integralen:

$$= \frac{1}{2}\left(\iint_{E_1} 3-x^2-y^2\,dx\,dy + \iint_{E_2\cup E_1} 4-3x^2-4y^2\,dx\,dy - \iint_{E_1} 4-3x^2-4y^2\,dx\,dy\right)$$

$$= \frac{1}{2}\left(\iint_{E_1} 2x^2+3y^2-1\,dx\,dy + \iint_{E_2\cup E_1} 4-3x^2-4y^2\,dx\,dy\right)$$

första integral: $x = \frac{1}{\sqrt{2}} r\cos t$, $y = \frac{1}{\sqrt{3}} r\sin t$, $r \in [0,1]$, $t \in [0,2\pi]$, $\frac{d(x,y)}{d(r,t)} = r\frac{1}{\sqrt{2}}\frac{1}{\sqrt{3}} = \frac{r}{\sqrt{6}}$

andra integral: $x = \frac{1}{\sqrt{3}} r\cos t$, $y = \frac{1}{2} r\sin t$, $r \in [0,2]$, $t \in [0,2\pi]$, $\frac{d(x,y)}{d(r,t)} = r\frac{1}{\sqrt{3}}\frac{1}{2} = \frac{r}{2\sqrt{3}}$

$$= \frac{1}{2}\left(\int_0^1\int_0^{2\pi}(r^2-1)\frac{r}{\sqrt{6}}\,dt\,dr + \int_0^2\int_0^{2\pi}(4-r^2)\frac{r}{2\sqrt{3}}\,dt\,dr\right)$$

$$= \frac{1}{2}\cdot 2\pi\left(\frac{1}{\sqrt{6}}\int_0^1 r^3-r\,dr + \frac{1}{2\sqrt{3}}\int_0^2 4r-r^3\,dr\right)$$

$$= \pi\left(-\frac{1}{4\sqrt{6}} + \frac{4}{2\sqrt{3}}\right) = \frac{\pi}{24}\left(16\sqrt{3}-\sqrt{6}\right)$$