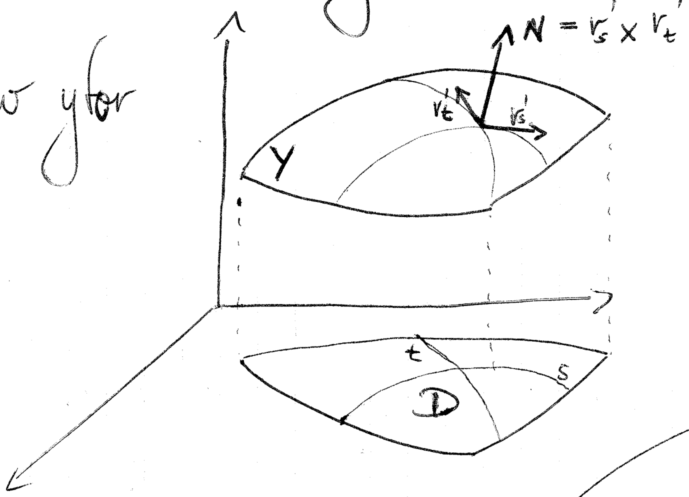
Räkneövning 15-04-23

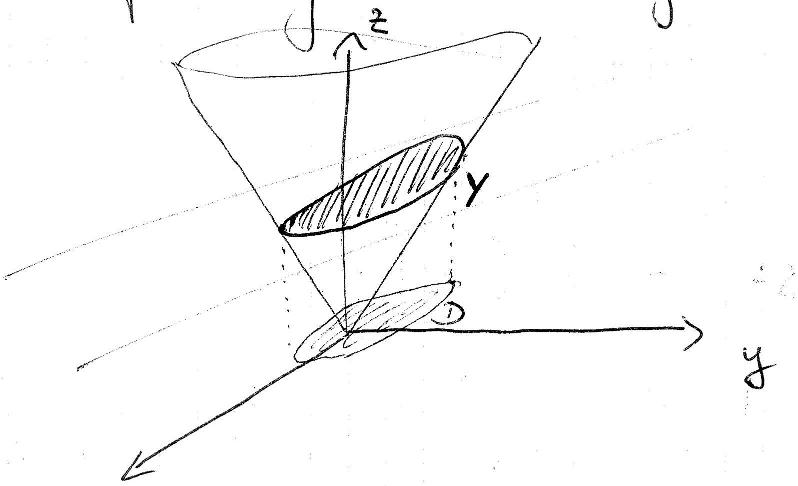
arean av ytor



transformation med funktionaldeterminant

$$\text{area}(Y) = \iint_Y 1 \, \underbrace{dS}_{\text{areaelement}} = \iint_D \underbrace{|r_s' \times r_t'|}_{\text{längden av normalvektorn till ytan}} \, ds\,dt$$

(28) arean av planets $z = 1 + x + 2y$ del som ligger innanför konytan $z = \sqrt{2x^2 + 10y^2}$



normalen till ytan:

direkt: $x + 2y - z = -1 \longrightarrow N = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$

parametrisering: $r(s,t) = (s, t, s + 2t + 1)$ $\qquad x = s, y = t$

$$r_s' = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad r_t' = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \quad r_s' \times r_t' = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$$

riktning spelar ingen roll, i båda fall: $\left\|\begin{pmatrix}1\\2\\-1\end{pmatrix}\right\| = \sqrt{1+4+1} = \sqrt{6}$

$$\iint_Y 1\, dS = \iint_D \sqrt{6}\, ds dt = \sqrt{6} \iint_D 1\, ds dt = \sqrt{6}\ \text{area}(\text{ellips } D)$$

D... ellips... projektion av Y på xy-planet:

skärningskurva: $1 + 2y + x = \sqrt{2x^2 + 10y^2}$

$$1 + 4y^2 + x^2 + 4y + 2x + 4xy = 10y^2 + 2x^2$$

$$x^2 - 2x + 6y^2 - 4y - 4xy = 1$$

$$(x^2 - 4xy + 4y^2) + 2y^2 - 2x - 4y = 1$$

$$(x-2y)^2 - 2(x-2y) + 2y^2 - 8y = 1$$

koordinatbyte: $s = x - 2y$, $t = y$ $\qquad \dfrac{d(s,t)}{d(x,y)} = \begin{pmatrix}1 & -2\\0 & 1\end{pmatrix}$

determinant: 1 $\Rightarrow$ arean stannar samma

$$s^2 - 2s + 2t^2 - 8t = 1$$

$$(s-1)^2 - 1 + 2(t-2)^2 - 8 = 1$$

koordinatbyte: $\xi = s - 1$, $\eta = t - 2$ $\qquad \dfrac{d(\xi,\eta)}{d(s,t)} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$ det 1

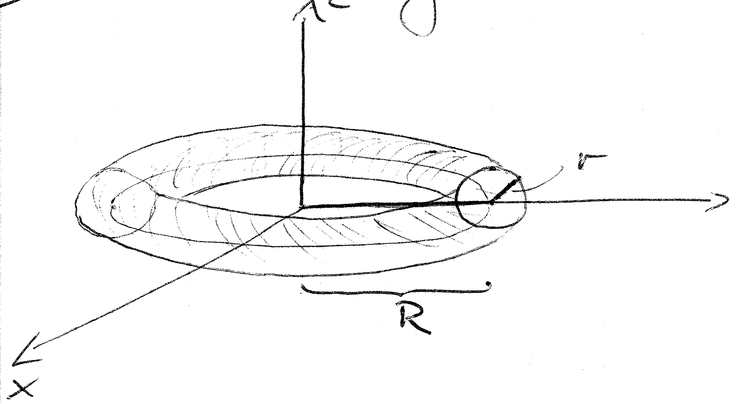
$$\xi^2 + 2\eta^2 = 10$$

$$\frac{\xi^2}{(\sqrt{10})^2} + \frac{\eta^2}{(\sqrt{5})^2} = 1 \qquad \rightsquigarrow \quad a = \sqrt{10} \quad b = \sqrt{5}$$

$$\text{area}(D) = a \cdot b \cdot \pi = \sqrt{10} \cdot \sqrt{5}\, \pi = 5\sqrt{2}\, \pi$$

$$\Rightarrow \text{area}(Y) = \sqrt{6}\, 5\sqrt{2}\, \pi = 10\sqrt{3}\, \pi$$

(27) $0 < r < R$ givna konstanter: arean av torus?



$$\begin{cases} x = (R + r\cos t)\cos s \\ y = (R + r\cos t)\sin s \\ z = r\sin t \end{cases}$$

$$0 \le s, t \le 2\pi$$

normalen: $r(s,t) = (x(s,t), y(s,t), z(s,t))$

$$r_s = (-(R+r\cos t)\sin s, (R+r\cos t)\cos s, 0)$$

$$r_t = (-r\cos s\sin t, -r\sin s\sin t, r\cos t)$$

$$r_s \times r_t = \begin{pmatrix} (R+r\cos t)r\cos s\cos t \\ (R+r\cos t)r\sin s\cos t \\ (R+r\cos t)r\sin^2 s\sin t + (R+r\cos t)r\cos^2 s\sin t \end{pmatrix} = (R+r\cos t)r \begin{pmatrix} \cos s\cos t \\ \sin s\cos t \\ \sin t \end{pmatrix}$$

% orientering spelar ingen roll men man kan se genom att sätta $s = t = 0$ att i punkten $r(0,0) = (R+r, 0, 0)$ pekar $r_s \times r_t(0,0) = (R+r)r\begin{pmatrix}1\\0\\0\end{pmatrix}$ i riktning av x-axeln $\Rightarrow$ utåt

$$|r_s \times r_t| = \sqrt{(R+r\cos t)^2 r^2 [\cos^2 s\cos^2 t + \sin^2 s\cos^2 t + \sin^2 t]} = (R+r\cos t)r$$

$$\text{area}(Y) = \iint_Y 1\,dS = \iint_D |r_s \times r_t|\,ds\,dt = \int_0^{2\pi}\int_0^{2\pi} (R + r\cos t)r\,ds\,dt$$

$$= \int_0^{2\pi}\int_0^{2\pi} Rr + r^2 \underbrace{\cos t}_{\substack{\text{försvinner efter} \\ \text{integration över en hel period}}}\,ds\,dt = 4\pi^2 rR$$