Exercise 1

$\alpha = \sqrt{2+\sqrt{5}}$

$\alpha^2 = 2 + \sqrt{5}$

$\alpha^4 - 4\alpha^2 + 4 = 5$

$\alpha^4 - 4\alpha^2 - 1 = 0$ candidate $x^4 - 4x^2 - 1 = p(x)$

monic and $p(\alpha) = 0$ we need to check irreducibility. It has no root (ratio test)

$(x^2 + ax + b)(x^2 + \alpha x + \beta) = x^4 - 4x^2 - 1$

$b\beta = -1 \Rightarrow b = -\beta = \pm 1$

$\beta + b + a\alpha = -4 \Rightarrow a\alpha = -4$

$b\alpha + \beta a = -\beta(a - \alpha) = 0$

$\beta \neq 0 \Rightarrow a = \alpha$

$\Rightarrow a^2 = -4$ impossible.

$\Rightarrow p(x) = x^4 - 4x^2 - 1$

$[\mathbb{Q}(\sqrt{2+\sqrt{5}}) : \mathbb{Q}] = 4 = \deg p(x)$

Exercise 2

$\alpha = \sqrt{2} + \sqrt{3}$

$\alpha^2 = 5 + 2\sqrt{6}$

$\alpha^4 - 10\alpha + 25 = 24$

$\alpha^4 - 10\alpha + 1 = 0$

candidate for minimal poly

$p(x) = x^4 - 10x^2 + 1$

monic & $p(\alpha) = 0$

we need to check irreducibility. The root test says that the possible roots are ± 1 & $p(\pm 1) \neq 0$

We write

$$x^4 - 10x^2 + 1 = (x^2 + ax + b)(x^2 + \alpha x + \beta)$$

$$b \cdot \beta = 1 \quad \Rightarrow \quad b = \beta = \pm 1$$

$$\beta a + \alpha b = 0 \quad \Rightarrow \quad \alpha + a = 0$$

$$\alpha = -a$$

$$b + \beta + a\alpha = -10$$

$b = 1$

$$2 + a\alpha = -10$$

$$a\alpha = -12$$

$$-a^2 = 12$$

impossible for $a \in \mathbb{Q}$

$b = -1$

$$-2 + a\alpha = -10$$

$$-a^2 = -8$$

impossible

So the minimal polynomial is

$$x^4 - 10x^2 + 1$$

$$\mathbb{Q}(\alpha) \subseteq \mathbb{Q}(\sqrt{2}, \sqrt{3}) \quad \text{since } \alpha = \sqrt{2} + \sqrt{3} \in \mathbb{Q}(\sqrt{2}, \sqrt{3})$$

we want to check that they are equal

this is the same as showing

$$[\mathbb{Q}(\sqrt{2},\sqrt{3}):\mathbb{Q}(\alpha)]=1$$

$$[\mathbb{Q}(\sqrt{2}\sqrt{3}):\mathbb{Q}]=[\mathbb{Q}(\sqrt{2}\sqrt{3}):\mathbb{Q}(\sqrt{2})]\cdot$$
$$\cdot[\mathbb{Q}(\sqrt{3}):\mathbb{Q}]$$

Now $[\mathbb{Q}(\sqrt{2}):\mathbb{Q}]=2$
since x^2-2 is the min polynomial

~~$[\mathbb{Q}(\sqrt{2}):\mathbb{Q}]$~~

$[\mathbb{Q}(\sqrt{2},\sqrt{3}):\mathbb{Q}(\sqrt{2})]\leq 2$ since $\sqrt{3}$ is a root of $x^2-3 \in \mathbb{Q}(\sqrt{2})[x]$

$$4\geq[\mathbb{Q}(\sqrt{2},\sqrt{3}):\mathbb{Q}]=[\mathbb{Q}(\sqrt{2},\sqrt{3}):\mathbb{Q}(\alpha)]\cdot$$
$$\cdot[\mathbb{Q}(\alpha):\mathbb{Q}]$$

$$=4[\mathbb{Q}(\sqrt{2},\sqrt{3}):\mathbb{Q}(\alpha)]$$

$$\Rightarrow 1\leq[\mathbb{Q}(\sqrt{2},\sqrt{3}):\mathbb{Q}(\alpha)]=1$$

alternatively find a basis of $\mathbb{Q}(\alpha)/\mathbb{Q}$ and express $\sqrt{2}$ and $\sqrt{3}$ as lc of the basis element. as done in class

$[\mathbb{Q}(\sqrt[n]{7},\sqrt[m]{7}):\mathbb{Q}]$

$$\leq n$$

$$= [\mathbb{Q}(\sqrt[n]{7},\sqrt[m]{7}):\mathbb{Q}(\sqrt[n]{7})][\mathbb{Q}(\sqrt[n]{7}):\mathbb{Q}]$$

$$= [\qquad]\cdot m \leq n\cdot m$$

$$= [\mathbb{Q}(\sqrt[n]{7},\sqrt[m]{7}):\mathbb{Q}(\sqrt[m]{7})]\cdot[\mathbb{Q}(\sqrt[m]{7}):\mathbb{Q}]$$

$$= [\qquad]\cdot n$$

$\Rightarrow n, m \mid [\mathbb{Q}(\sqrt[n]{7},\sqrt[m]{7}):\mathbb{Q}]$

$\gcd(n,m)=1$

$\Rightarrow n\cdot m \mid [\mathbb{Q}(\sqrt[n]{7},\sqrt[m]{7}):\mathbb{Q}] \leq n\cdot m$

$\Rightarrow [\mathbb{Q}(\sqrt[n]{7},\sqrt[m]{7}):\mathbb{Q}] = n\cdot m$

Exercise 3

$p(x)$ is irreducible ~~s~~ince has no roots and $\deg p(x) \leq 3$

$(p(x))$ is prime & $\neq (0)$

$\mathbb{Z}_2[x]$ is PID $\Rightarrow (p(x))$ is max

$\Rightarrow$ ~~(p(x~~ $\mathbb{Z}_2[x]/(p(x))$ field

$p(\alpha)=0$ (done in class)

$\Rightarrow p$ is the minimal poly of α

Basis $1 \quad \alpha \quad \alpha^2$

$$\alpha^6 = (\alpha^3)^2 = (\alpha+1)^2 = \alpha^2+1$$

$$\begin{array}{l|l} x^3+x+1 & x+1 \\ \hline x^3+x^2 & x^2+x \\ \hline /\!/ \quad x^2+x+1 & \\ \quad\ x^2+x & \\ \hline \qquad 1 & \end{array}$$

$$x^3+x+1 = (x^2+x)(x+1)+1$$

$$(x^2+x)(x+1) \equiv 1 \mod p(x)$$

$$\boxed{\alpha^2+\alpha}$$

The other is the same only $\neq$ computatio

Exercise 4

① it is $x^2 - 2$

elements of $\mathbb{Q}(\sqrt[3]{3})$ are

$$a + b\sqrt[3]{3} + c\sqrt[3]{9} \quad \text{with } a,b,c \in \mathbb{Q}$$

Check that these are not roots

(2) $\alpha = \sqrt{2+\sqrt[3]{3}}$ eliminate $\sqrt{\ }$

find α as a root of a poly.

$\mathbb{Q}(\alpha) = \mathbb{Q}(\beta)$

$$\deg p_\alpha = [\mathbb{Q}(\alpha) : \mathbb{Q}] =$$
$$[\mathbb{Q}(\beta) : \mathbb{Q}] = \deg p_\beta$$

(4) same as before

(5) same as befor