TM decidable & recognizable languages.

Language L = set of strings over alphabet Σ

L TM-recognizable, if $\exists$ TM M that accepts w $\forall w \in L$

more precise: if $L = \{ w \mid M$ reaches accept state on $w \}$

& thus, $\forall w \notin L$ M rejects & halts OR does not halt.

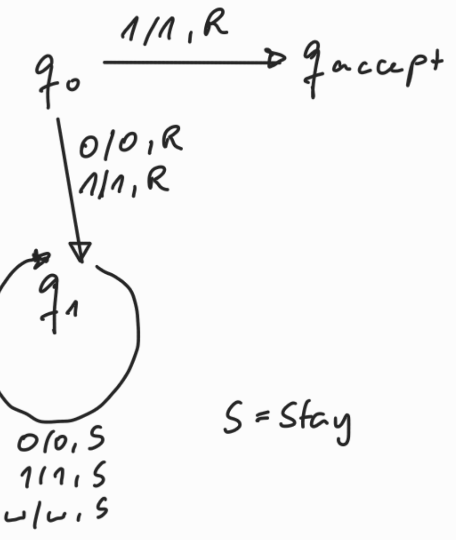
L is decidable if $\exists$ TM M st $\forall w \in L$: M accepts w
& $\forall w \notin L$: M rejects w

[in lecture: L decidable if $\exists$ TM M st $\forall w \in L$: M halts
& $w \in L \Leftrightarrow M$ accepts w]

Exampl

consider $L = \{ w \mid w \in \{0,1\}^*,$ 1st letter of w is $1 \}$.

& TM M:



S = Stay

$\Rightarrow$ M recognizes L
but for $w \notin L$
M does not halt.

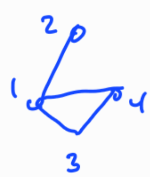
Solving a problem can be viewed as recognizing a language:

<u>Exampl:</u> Decision problem D: Is longest shortest path (= diam) of graph G < k ?

$\left(diam(G) = \max_{u,v \in V} \{dist(u,v)\} < k ? \right)$

D as language for $G = (V, E)$ & $k \mathrel{\hat{=}}$ all words of the form $[V(G); E(G); k]$ where the answer is YES.

(1)  & $k = 1$ → $[1,2 ; (1,2); 1]$ → Answer NO

(2)  & $k = 3$ → $[1,2,3,4; (12),(14),(43),(31); 3]$ → Answer: YES

⇒ solving D is equivalent to recognize all yes Instances of the corresponding language.

IF TM M accepts $[V(G); E(G); k]$ → <u>yes</u>
ELSE → NO.

if in ∞-loop does not help. [not poly time]

TM polytime = TM that halts after a polytime nr of steps.

$\mathbf{P} = \{ L \subseteq \{0,1\}^* \mid \exists$ TM that recogn. L in poly time (always halts) ⟺ decides L. $\}$

⇒ Solving dec. prob in poly time ≙ Decide corresp. lang in poly time!

Recognizable languages that are not decidable

Consider following language:

$L_{HALT} = \{(m, w) : m, w \in \{0,1\}^* \ \&$ m is description of TM M that accepts w (& HALTS) $\}$.

also called "universal language"

This language is TM recognizable but not decidable.

WHY?
- L_{HALT} is TM-recognizable by "universal" TM [not shown here].
- L not decidable:

Assume, for contradiction, that L_{HALT} is decidable.

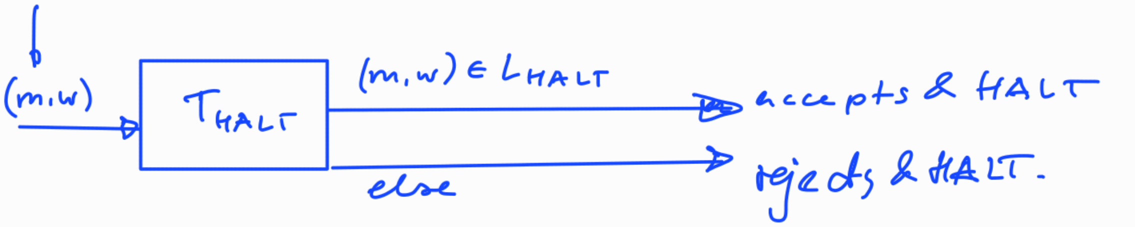
$\Rightarrow \exists$ TM T_{HALT} st $\forall (m,w) \in L_{HALT}$: T_{HALT} accepts, halt

& $\forall (m,w) \notin L_{HALT}$ T_{HALT} rejects, halt

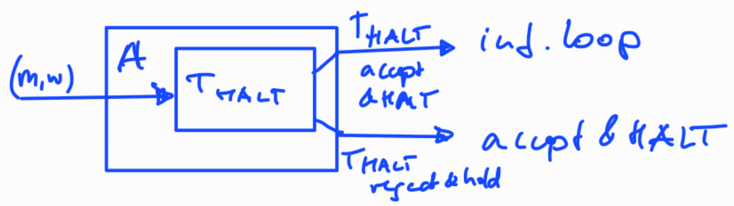
[$\Leftrightarrow$ HALTING PROBLEM ↗ Part 2 complexity

Here as TM:]

m of TM & m accept w.



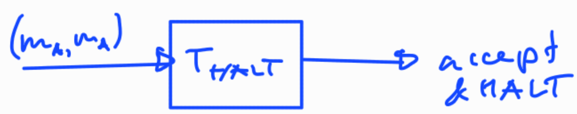
Now construct TM A:



consider (m_A, m_A) where $m_A \in \{0,1\}^*$ is description of A

2 cases:

(1) $(m_A, m_A) \in L_{HALT} \Rightarrow$



but then A (described as m_A) does not accept

$\Rightarrow (m_A, m_A) \notin L_{HALT}$ ↯

(2) $(m_A, m_A) \notin L_{HALT} \Rightarrow$

but then A accepts & HALT

$\Rightarrow (m_A, m_A) \in L_{HALT}$ ↯

$\square$

languages that are not recognizable

Thm: There are languages that are not TM-recognizable.

Proof: need to show that $\exists L \subseteq \Sigma^*$ st $\forall$ TM M: M does not accept some $w \in L$.

non-selfacceptable language $NSA := NSA_1 \cup NSA_2$.

$NSA_1 = \{ w : w \in \{0,1\}^*, \underline{w}$ is "description" of TM M & M does not accept $w \}$

= set of all words over 0,1 that describe a TM M which does not accept w.

$NSA_2 = \{ w : w \in \{0,1\}^*, w$ does not describe any TM$\}$.

recap:

$L_{HALT} = \{(m, w) : m, w \in \{0,1\}^*$ & m is description of TM M that accepts w (& HALTS) $\}$.

$\Rightarrow \overline{L_{HALT}} \mathrel{\hat{=}} NSA := NSA_1 \cup NSA_2$

Observe $NSA \neq \emptyset$ since
$NSA_1 \neq \emptyset$ $\left(\begin{array}{l} w \text{ is TM that accepts only string "0"} \\ |w| > 1 \Rightarrow \text{TM does not accept } w \end{array} \right)$

& $NSA_2 \neq \emptyset$ since if $w = \{0\}$
$\Rightarrow |w| = 1 \Rightarrow w$ cannot describe a TM
$\Rightarrow w \in NSA_2$

Assume now, for contradiction that NSA is TM-recognizable.

$\overset{\text{Def}}{\Rightarrow}$ $\exists$ TM M_0 that accepts all $w \in NSA$. [accepts & halts $\forall w \in NSA$]

let $w_0 \in \{0,1\}^*$ be a description of M_0.

Can M_0 accept w_0?

Assume yes.: $NSA = \{w \mid M_0 \text{ reaches accept. state on } w\}$

Since M_0 accepts $w_0 \Rightarrow w_0 \in NSA = NSA_1 \cup NSA_2$

Since w_0 describes TM $\Rightarrow |w_0| > 1 \Rightarrow w_0 \notin NSA_2$

$\Rightarrow w_0 \in NSA_1 = \{w \mid w \text{ desc. TM that does Not accept } w\}$.

Since $w_0 \in NSA_1 \Rightarrow M_0$ does not accept w_0 ↯

$\Rightarrow$ M_0 cannot accept w_0

$\Rightarrow$ by definition $w_0 \in NSA_1 \subseteq NSA$

$\Rightarrow$ $\exists\, w_0 \in NSA$ that is not accepted by M_0

$\Rightarrow$ NSA is not recognizable by M_0 ↯

$\square$