Algorithms and Complexity

1./2. crash course - recap - time complexity

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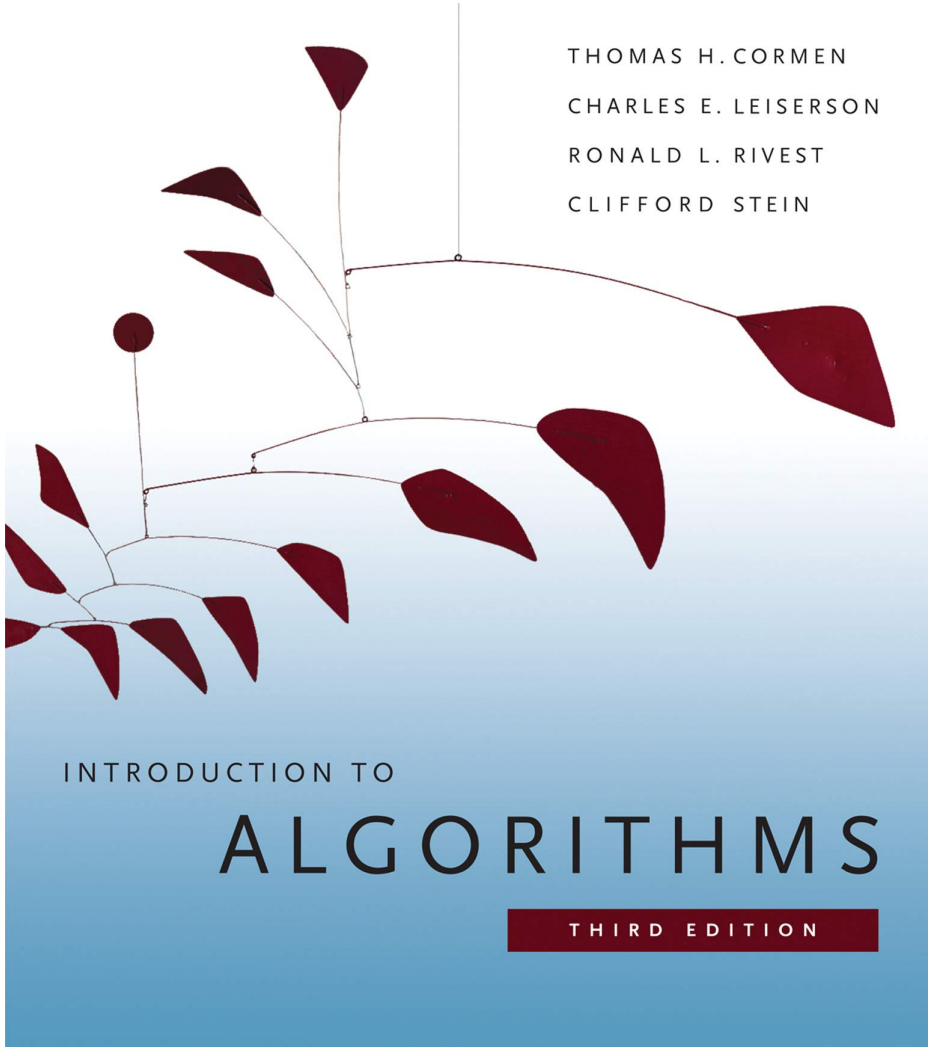
Time complexity



Chapter 1-3 and 4.5

Time complexity

Runtime of an algorithm

Naive idea: measure the time from start to end in (mili)seconds

say we want to know for some input N how fast the algorithm is:

- $N = 4000$ and runtime 6.3 seconds
- $N = 8000$ and runtime 51.1 seconds
- $N = 16000$ and runtime 410.8 seconds

Hypthesis: For arbitrary N runtime is $\sim 10^{-10} N^3$

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Time complexity

NOT: measure runtime on a specific computer

BUT: determine effort for idealized computer model
(e.g. Random-Access-Maschine (RAM-model))

Need abstract measure for complexity to estimate asymptotic costs that depends on the size of the input

Time complexity

Add two numbers

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+		2	8	3	4
=	1	3	9	7	1

Takes 5 single additions.

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The RAM-model contains instructions commonly found in real computers:

- arithmetic (such as add, subtract, multiply, divide, remainder, floor, ceiling),
- data movement (load, store, copy), and
- control (conditional and unconditional branch, subroutine call and return).

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- arithmetic (such as add, subtract, multiply, divide, remainder, floor, ceiling),
- data movement (load, store, copy), and
- control (conditional and unconditional branch, subroutine call and return).

Each such instruction is counted as one time-unit and thus, takes a constant amount of time.

Hence, we essentially count the number of execution of instructions (as the number of operations)

Time complexity

The runtime of an algorithm with input I is denoted by $T(|I|)$, where $|I|$ is the size of the input and $T(|I|)$ is the number of operations/instructions used in this algorithm with input I.

Input $I = A[n]$, input size $|I| = n$

COUNT_ZEROS(array $a[n]$)

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1: int count = 0
2: for (int i=0; i<n; i++) do
3:     if a[i] == 0 then
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We are, in general, not interested in specific values for n but the asymptotic behaviour of $T(n)$ (that is for large n)

Time complexity

Big-O-, Big-Θ- and Big-Ω-Notation

The asymptotic complexity $T(n)$ is limited from above by a function $f(n)$ whenever there are postive constants n_0 and c such that for all $n > n_0$ it holds that

$$T(n) \leq cf(n).$$

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The notation $T(n) = O(f(n))$ is also very commonly used.

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$T(n) = 5n + 6$ is in $O(n)$:

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$T(n) = 5n+6$ is in $O(n)$:

$5n \leq 5n$ for all n (incl. $n \geq 1$)

$6 \leq 6n$ for all $n \geq 1$

Thus, $T(n) = 5n+6 \leq 5n+6n = 11n = cf(n)$ for all $n \geq 1$

Thus, $5n+6 \in O(n)$ (choose $c = 11$ and $n_0 = 1$.)

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Note, $T(n) = 5n+6 \leq 11 \cdot n \leq 11 \cdot n \log n \leq 11 \cdot n^{100}$

Hence, $T(n) \in O(n \log n)$, $T(n) \in O(n^{100})$. We usually want to find tight upper bounds.

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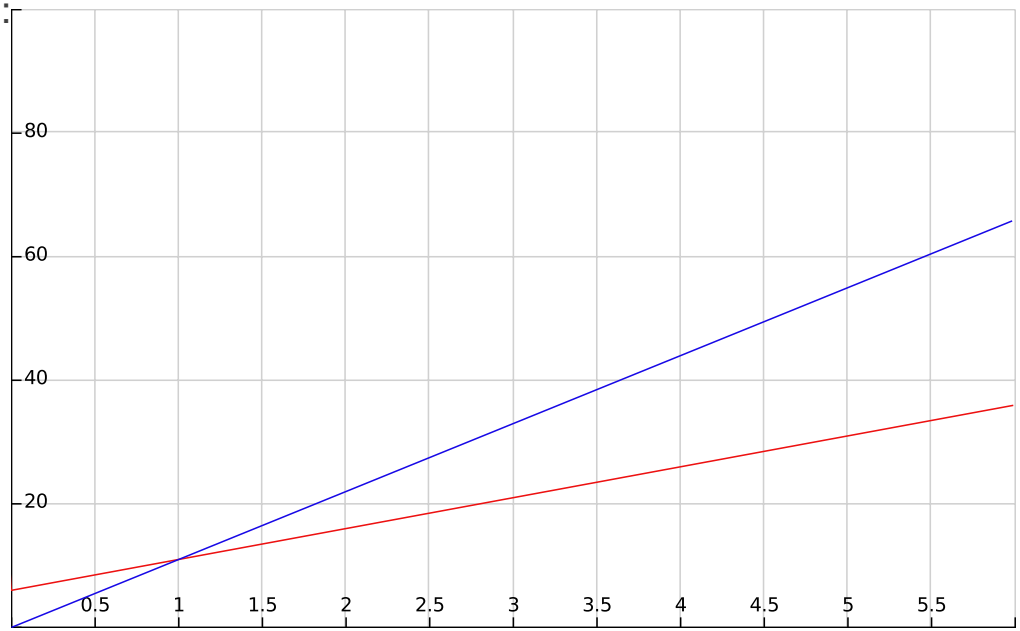

5n+6 and 11n

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$$T(n) = \sum_{i=0}^{p} x_i n^i = x_p n^p + \ldots + x_1 n^1 + x_0 n^0 \in O(n^p) \text{ for all } n \geq 1$$

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$$T(n) = \sum_{i=0}^{p} x_i n^i \leq \sum_{i=0}^{p} M n^i = M(\sum_{i=0}^{p} n^i)$$

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$$M(\sum_{i=0}^{p} n^i) \leq M(\sum_{i=0}^{p} n^p) = M \cdot ((p+1) \cdot n^p) = c \cdot n^p \text{ for } c = M \cdot (p+1) \text{ and all } n \geq n_0 = 1$$

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Thus, choose $c = M \cdot p$ and $n_0 = 1$ to conclude that $T(n) \in O(n^p)$.

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In a similar way, one can compute lower bounds $f(n)$ to show that $T(n)$ grows (asymptotically) at least as “fast” as $f(n)$

$$T(n) \in \Omega(f(n)) :\Leftrightarrow \exists c > 0, n_0 > 0 : \forall n > n_0 : T(n) \geq cf(n)$$

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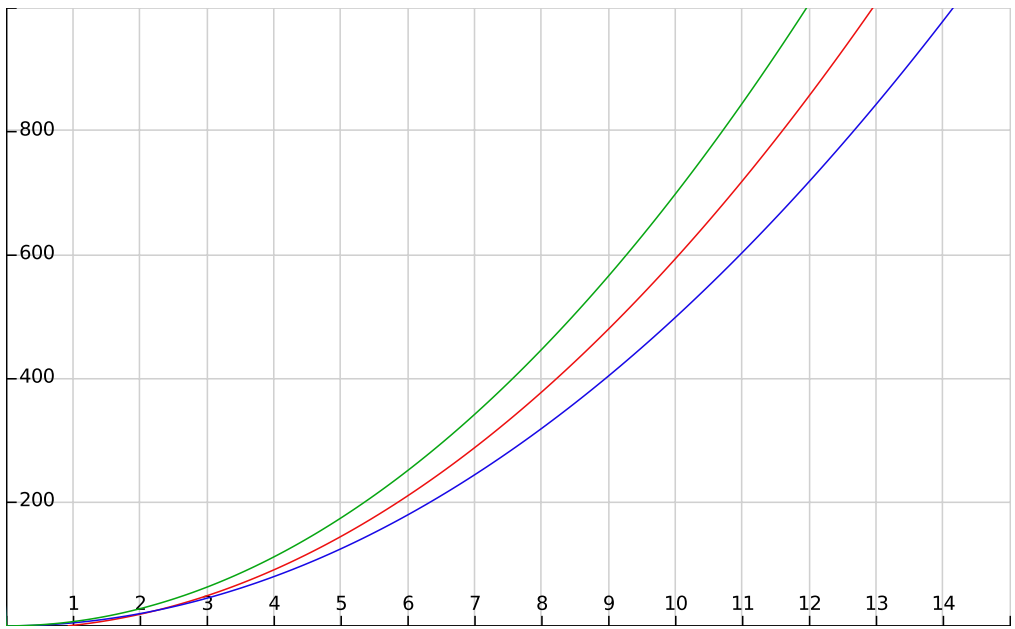
Example: $T(n) = 6n^2 - 5 \in \Omega(n^2)$ (choose $c = 5$ and $n_0 = 3$)



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Hence, $T(n) \in \Omega(n^2)$ and $T(n) \in O(n^2)$

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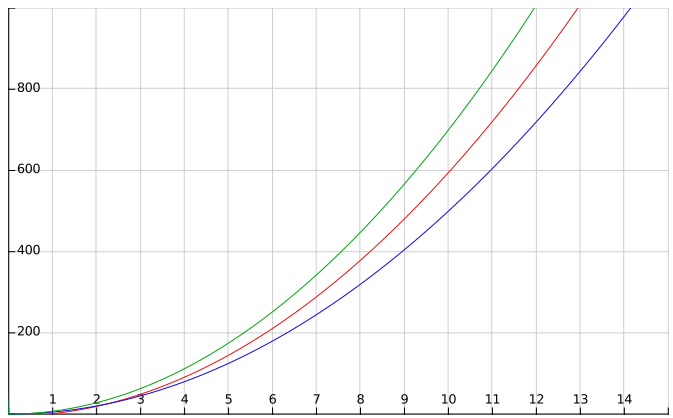
$T(n) = 6n^2 - 5 \in O(n^2)$ (choose e.g. $c = 7$ and $n_0 = 1$)



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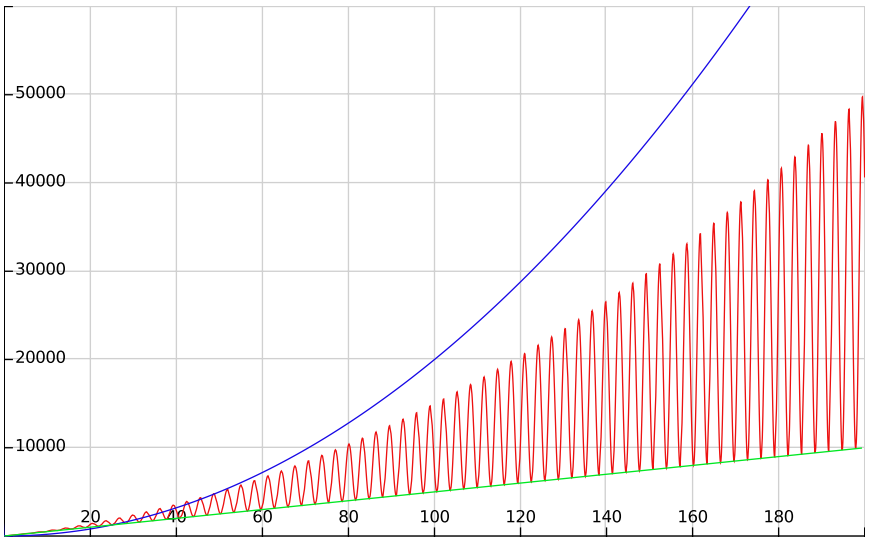

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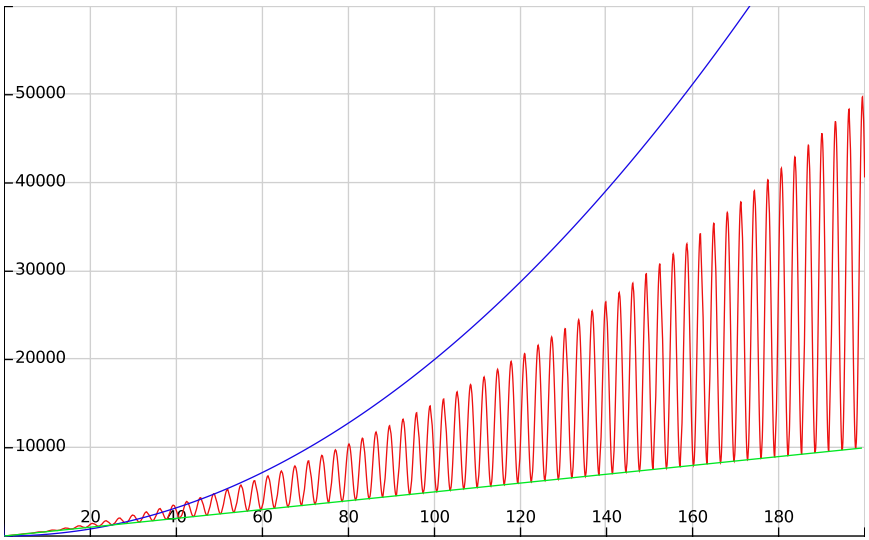
Example: $T(n) = n^2(sin(n))^2 + 50n$

Since $sin(n) \leq 1$, we have $n^2(sin(n))^2 + 50n \leq n^2 + 50n \leq 2n^2$ for all $n \geq 50$ and thus, $T(n) \in O(n^2)$



$50n \leq n^2 sin(n) \leq x^2$

Time complexity

Big-O-, Big-Θ- and Big-Ω-Notation

$$T(n) \in O(f(n)) :\Leftrightarrow \exists c > 0, n_0 > 0 : \forall n > n_0 : T(n) \leq cf(n)$$
$$T(n) \in \Omega(f(n)) :\Leftrightarrow \exists c > 0, n_0 > 0 : \forall n > n_0 : T(n) \geq cf(n)$$

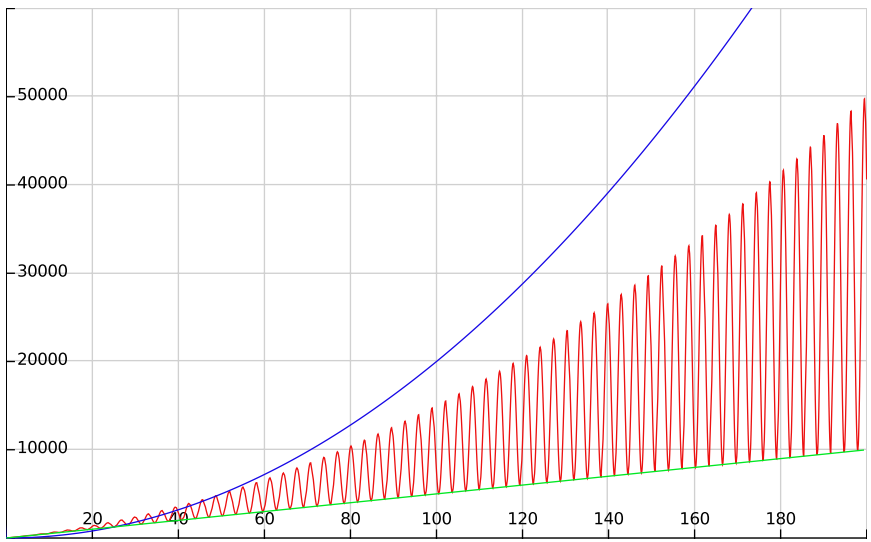
If $T(n) \in O(f(n))$ and $T(n) \in \Omega(f(n))$ then $T(n) \in \Theta(f(n))$

Example: $T(n) = n^2(sin(n))^2 + 50n$

Since $sin(n) \leq 1$, we have $n^2(sin(n))^2 + 50n \leq n^2 + 50n \leq 2n^2$ for all $n \geq 50$ and thus, $T(n) \in O(n^2)$

Since $n^2(sin(n))^2 \geq 0$, we have $n^2(sin(n))^2 + 50n \geq 50n$ for all $n \geq 1$ and thus, $T(n) \in \Omega(n)$

(and we can't do better - thus "no Θ for $T(n)$"!)



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O-Notation

$O(\dots)$ (rt=runtime)	typical framework	typical examples
$O(1)$ constant rt	`a=b+c` // `if(a<b)`	assignments, in/output, 32/64bit-arithmetic, cases
$O(\log n)$ logarithmic rt	`while(N>1) N = N/2`	binary search
$O(n)$ linear rt	`for(i=0; i<n; i++){...}`	loop find the maximum
$O(n^2)$ quadratic rt	`for(i=0; i<n; i++)` `  for(j=0; j<n; j++) {...}`	double loop, check all pairs

$O(n^3)$ cubic rt	`for(i=0; i<n; i++)`
	`  for(j=0; j<n; j++)`
	`  for(k=0; k<n; k++) {...}`
		triple loop,
		check all triples

$O(2^n)$ exponential rt	`see combinatorial lecture;)`	exhaustive search check all subsets

Time Complexity

General rules

Now, O-notation only which is the most interesting part for us.

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Loops: Product of number of runs and costs of most expencive task within this loop

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Summation of parts: If algorithm A consists of two independent parts A_1 and A_2 (with runtime $T_1(n) \in O(f(n))$, resp., $T_2(n) \in O(g(n))$), then complexity of A is

$$T(n) = T_1(n) + T_2(n) = O(f(n) + g(n)) = O(\max\{f(n), g(n)\})$$

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DO_SMTH(int n)

```
1: print "Hello World"
2: for (int i = 0; i < n; i++) do
3:     if n is even then
4:         return 0
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6:         for (int j = 0; j < n; j++) do
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All assignments, cases, statements (eg. **print**, **int** $i = 0$, $n = n \cdot n$, **return** 0, $j < n$) in $O(1)$ time

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DO_SMTH consists of two main-parts:

A_1 = **print** "Hello World" and A_2 = Line 2-7

Hence, runtime of DO_SMTH is $O(1)+$ runtime A_2 $\Longrightarrow$ examine A_2 !

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number of runs = n and most expensive task "$n = n \cdot n$" in $O(1)$ time

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Thus, runtime DO_SMTH is $O(1)+$ runtime $A_2 \in O(1+n^2) = O(n^2)$

Time Complexity

Further example

DO_SMTH(graph $G = (V, E)$)

```
1: for (all vertices v ∈ V) do
2:     for (all vertices x ∈ N(v)) do
3:         print "neighbor of v is x"
```

Time Complexity

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Naive way:

$T(n) = |V|^2$ since for all $|V|$ vertices, we print all neighbors in $N(v)$ and $|N(v)| \leq |V|$

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For each $v \in V$ we print all $\deg(v)$ vertices.

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For each $v \in V$ we print all $\deg(v)$ vertices.

Hence, $T(n) = \sum_{v \in V} \deg(v) = 2|E| \in O(|E|)$.

Thus, instead of a quadratic runtime $O(|V|^2)$ this algorithm has even linear runtime $O(|E|)$

Time Complexity

Further example

HALVE(int n)

1: **while** $(n > 1)$ **do**
2: $\quad n = \frac{n}{2}$ *(pre-decimal point position)*

Time Complexity

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$$T(n) = T(1) + T(\frac{n}{2})$$

Time Complexity

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$$T(n) = T(1) + T(\frac{n}{2})$$
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Hence, we ask: When is "input for while" $\frac{n}{2^N} \leq 1$? Answer: $\frac{n}{2^N} \leq 1 \iff n \leq 2^N \iff log_2(n) \leq N$

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Put $N = \log_2(n)$

$$
\begin{aligned}
T(n) &= NT(1) + T(\frac{n}{2^N}) \\
&= \log_2(n)T(1) + T(1) \\
&\in O(\log_2(n))
\end{aligned}
$$

Time Complexity

Further example

// SUM returns the sum $\sum_{i=1}^{n} i$, where $n \geq 1$.
SUM(int n)

```
1: if (n = 1) then
2:     return 1
3: return n + SUM(n − 1)
```

$$
\begin{aligned}
T(n) &= T(1) + T(n-1) \\
&= T(1) + (T(1) + T(n-2) = 2T(1) + T(n-2) \\
&= \ldots \\
&= (n-1)T(1) + T(1) \in O(n) \text{ since } T(1) \in O(1)
\end{aligned}
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Example: Fibonacci Numbers $1,1,2,3,5,8,13,21,34,\ldots$

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naive recursive way (there are more efficient algorithms (dynamic programming)):

FIB(int $n \geq 1$)

1: **if** $n \leq 2$ **then** $f = 1$
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$$\begin{aligned} T(n) &= \Theta(1) + T(n-1) + T(n-2)) \\ &\geq 2T(n-2) = 2(T(n-3) + T(n-4)) \end{aligned}$$

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Hence, FIB has exponential runtime.

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Master Theorem

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Let $a \geq 1$ and $b > 1$ be constants and let $T(n)$ be a function over the positive numbers defined by the recurrence

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Examples: Note $T(1) = \Theta(1)$.

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- Exmpl HALVE: $T(n) = T(\frac{n}{2}) + \Theta(1)$. Here, $a = 1, b = 2, n^d = 1$ and thus, $d = 0$. We have $a = 1 = 2^0 = b^d$ and thus, $T(n) = \Theta(1 \log_2 n) = \Theta(\log_2 n)$

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$$T(n) = aT(n/b) + \Theta(n^d).$$

Then,

$$T(n) = \begin{cases} \Theta(n^d) & \text{if } a < b^d \\ \Theta(n^d \log_2 n) & \text{if } a = b^d \\ \Theta(n^{\log_b(a)}) & \text{if } a > b^d \end{cases}$$

Examples: Note $T(1) = \Theta(1)$. Put d=2, b=3

a=8 $T(n) = 8T(\frac{n}{3}) + \Theta(n^2) \implies T(n) = \Theta(n^2)$

a=9 $T(n) = 9T(\frac{n}{3}) + \Theta(n^2) \implies T(n) = \Theta(n^2 \log_2 n)$

a=10 $T(n) = 10T(\frac{n}{3}) + \Theta(n^2) \implies T(n) = \Theta(n^{log_3(10)})$

Space Complexity

Space complexity is a measure of the amount of working storage an algorithm needs.

Similar to time complexity, space complexity is often expressed asymptotically in big-O notation

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Requires 3 units of space for the parameters x, y, z and 1 for the local variable r.

Space complexity is in $O(1)$

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SUM(array a of length n)

```
1: int r = 0
2: for (int i = 0; i < n; ++i) do
3:     r = r + a[i];
4: return r
```

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1: **int** $r = 0$
2: **for** (**int** $i = 0; i < n;$ ++i) **do**
3: $\quad r = r + a[i];$
4: **return** r

Requires n units of space for array a and 2 for the local variables r and i.

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But be careful here: If things are passed by pointer or reference, then space is shared.