Algorithms and Complexity

2. Complexity

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Reminder: O-, Θ- and Ω-Notation

For positive functions f and g, we define

- $g(n) \in O(f(n)) :\Leftrightarrow \exists c > 0, n_0 > 0 : \forall n > n_0 : g(n) \leq cf(n)$
- $g(n) \in \Omega(f(n)) :\Leftrightarrow \exists c > 0, n_0 > 0 : \forall n > n_0 : g(n) \geq cf(n)$
- $g(n) \in \Theta(f(n)) :\Leftrightarrow g(n) \in O(f(n))$ and $g(n) \in \Omega(f(n))$.

The notation $g(n) = O(f(n))$ is also very commonly used.

WHITEBOARD: merge sort vs insertion sort $\Longrightarrow$ runtime matters!

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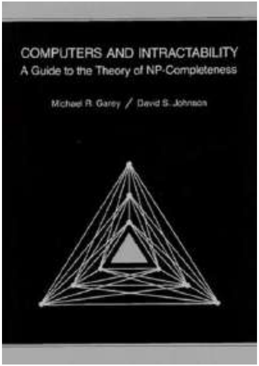
A1 WHITEBOARD: unsolvable (undecidable) problem

- the halting problem
- almost all decision problems cannot be solved by algorithms.

A2 Theory of NP-completeness

Theory of NP-completeness: Literature



Computers and Intractability: A Guide to the Theory of NP-Completeness
by Michael R. Garey, David S. Johnson
Published January 15th 1979 by W. H. Freeman

In a nutshell:

Aim: How difficult or easy is a problem that we want to solve?

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  $\Rightarrow$ Should we search further for polynomial-time algorithms?

  Or could we stop searching, since there is some evidence that no polynomial-time algorithm may exist?

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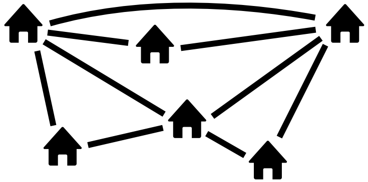

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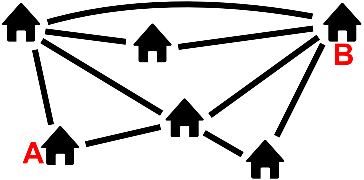

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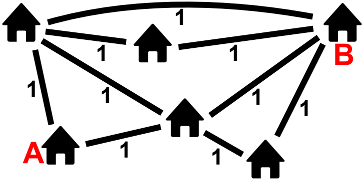

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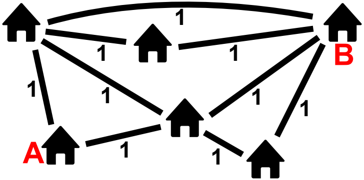

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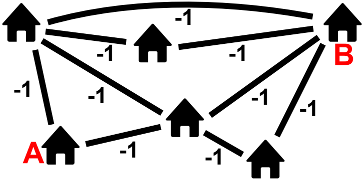

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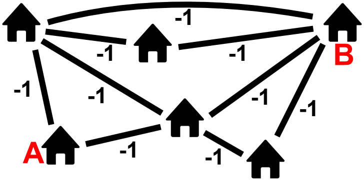

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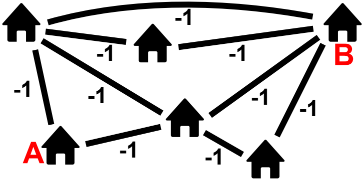

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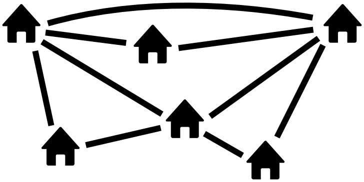

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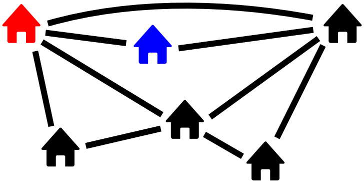

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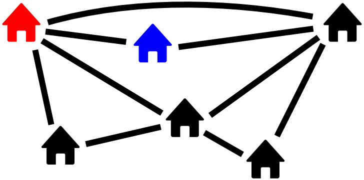

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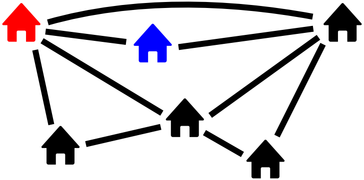

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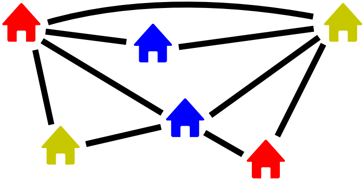

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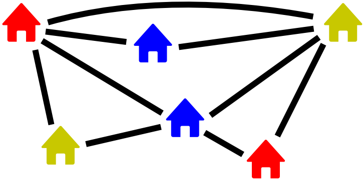

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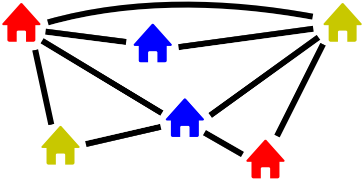

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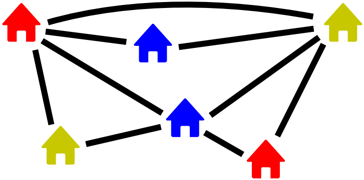

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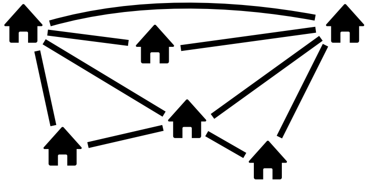

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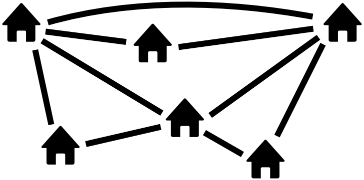

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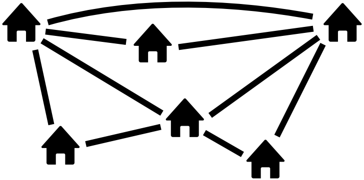

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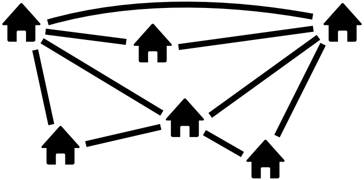

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What are these assumptions?

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- Main Ingredients
  - Optimization problems vs. decision problems
  - Classes P and NP
  - Reduction and NP-hardness
  - NP-completeness

We start with a brief overview of the main ingredients to answer these questions.

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- Decision problems have a very natural, formal counterpart called "language", as we have seen in the section about "TM and languages that are accepted by TM"

Outline

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  - Classes P and NP
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TM and Algorithms

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The TMs considered so-far are deterministic, i.e., each step of computation is uniquely determined by the transition function δ and the same input will always yield the same computational steps.

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NP means "**N**on-deterministic **P**olynomial" (**NOT:** *non-polynomial*!)

Keep in mind that algorithms are essentially defined in terms of TM.

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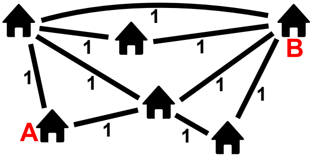
Example: Shortest Path



Question: Does there exist a path of length ≤ 4 from A to B?

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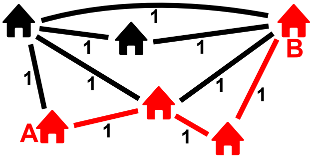

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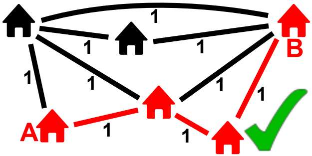

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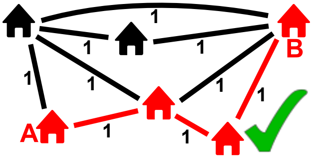

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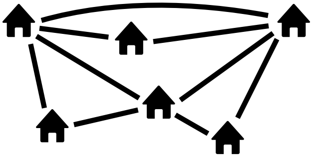
Example: 3-coloring



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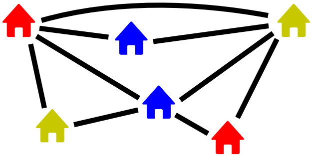

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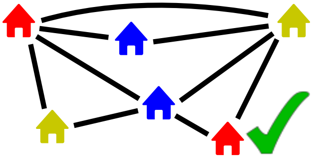

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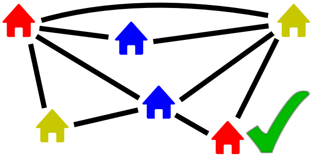

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The formal theory of P and NP is defined in term of languages and Turing machines

Outline

- Main Ingredients
  - Optimization problems vs. decision problems ✓
  - Classes P and NP ($P \subsetneq NP$)✓
  - Reduction and NP-hardness
  - NP-completeness

Reduction . . .

. . . is about transforming one problem into another problem

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ARRR, we leave you at this lonely island if you cannot answer the following question correctly within 2 minutes:

Are there less then 12222 coins in the chest?

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= 12220 * 8g

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Answer: YESSSS!!

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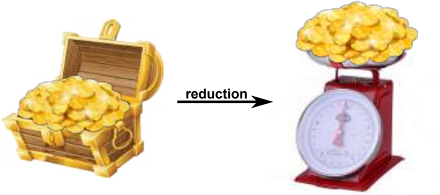

We reduced a counting problem to a weighting problem!

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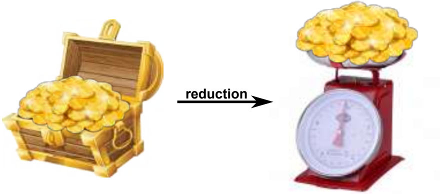

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D, D' decision problems

D' (number coins)
IN: Set of coins & integer K
Q: Number of coins $\leq K$?

D (weight of items)
IN: Item i & integer L
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Reduction . . .

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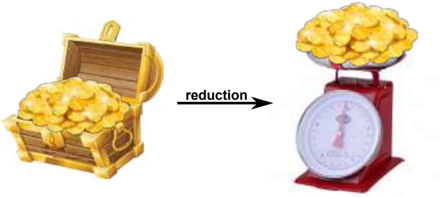

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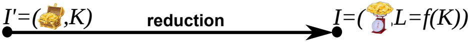

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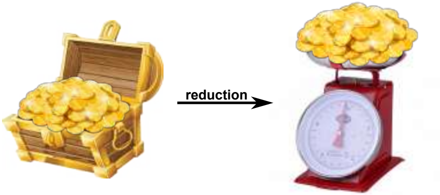

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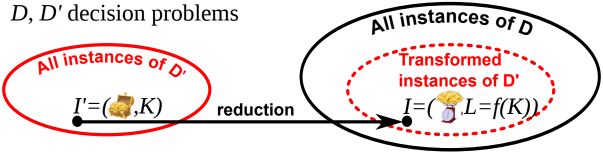

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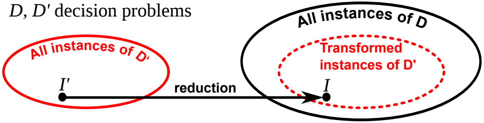

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Reduction

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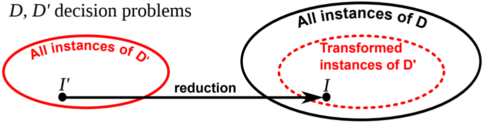

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A **reduction** from D' to D is a procedure that transforms every instance I' of D' to an instance I of D such that

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Reduction

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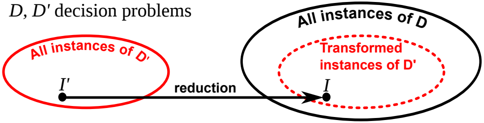

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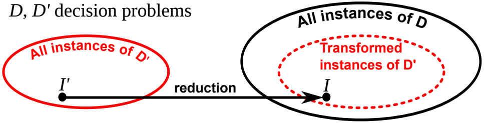

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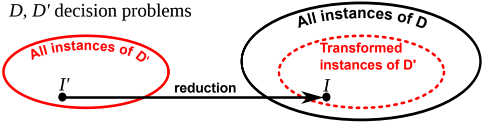

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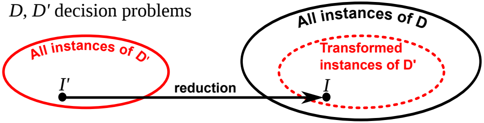

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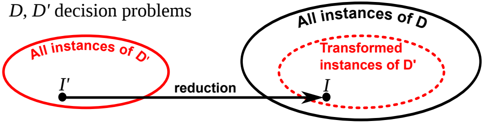

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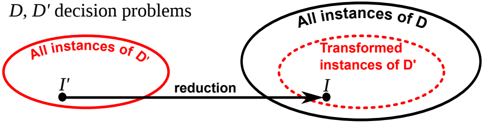

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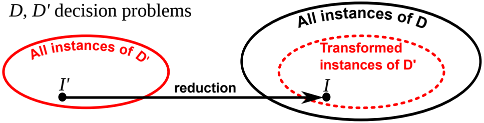

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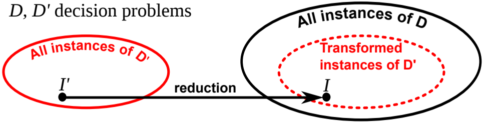
- it is reasonable to assume that for D there are no polynomial-time algorithm.

Reduction and NP-hard

D, D' decision problems



$\mathrm{P} \;= \{$dec. probl. *solvable* in polynomial time$\}$
$\mathrm{NP} = \{$dec. probl. *verifiable* in polynomial time$\}$

instance = *specified input*.

A **reduction** from D' to D is a procedure that transforms every instance I' of D' to an instance I of D such that

- the transformation can be done in *polynomial time* (= "easy") and
- I has YES-answer if and only if I' has YES-answer.

$\Rightarrow$ Every algorithm that solves D can be used to solve D'.

A dec. problem D is **NP-hard** if *every* problem $D' \in \mathrm{NP}$ can be reduced to D.

If D is NP-hard, then

- every problem $D' \in \mathrm{NP}$ can be considered as a "special case" of D.
- D is at least as difficult to solve as any other problem in NP.
- D is at least as difficult to solve as those problems in NP for which no polynomial-time algorithm exists ($P \subset NP$)
- it is reasonable to assume that for D there are no polynomial-time algorithm.

(Q1) What does **difficult** formally mean? **Answer:** NP-hard

Outline

- Main Ingredients
  - Optimization problems vs. decision problems ✓
  - Classes P and NP ($P \subsetneq NP$)✓
  - Reduction and NP-hardness ✓
  - NP-completeness

NP-complete

(Q1) What does **difficult** formally mean? **Answer:** NP-hard

(Q2) Are there problems for which no polynomial-time algorithm exists *under reasonable assumptions*? **(= intractable problems)**

What are these assumptions? **Answer:** $\mathbf{P} \subset \mathbf{NP}$

NP-complete

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(Q2) Are there problems for which no polynomial-time algorithm exists *under reasonable assumptions*? **(= intractable problems)**

Answer: Yes, NP-complete problems

What are these assumptions? **Answer:** $\mathbf{P} \subset \mathbf{NP}$

A decision problem D is **NP-complete** if

- $D \in \mathrm{NP}$
- D is NP-hard: *every* problem in NP can be reduced to D.

NP-complete

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(Q2) Are there problems for which no polynomial-time algorithm exists *under reasonable assumptions*? **(= intractable problems)**

Answer: Yes, NP-complete problems

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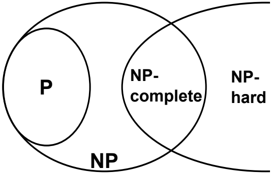

NP-complete problems are the most difficult problems in NP.

NP-complete

Note, Languages and Decision problem can be seen as being "equivalent", i.e., in the following definition, we can use these terms interchangeably.

A decision problem D is **NP-complete** if

- $D \in \mathrm{NP}$
- D is NP-hard: *every* problem in NP can be reduced to D.

  In symbols,

$$\forall D' \in \mathrm{NP} \colon D' \preceq_p D$$

Q: How to show that EVERY problem in NP can be reduced to D?

A: There was a first problem "SAT" that was shown to be NP-complete:
Cook-Levin-Thm 1971 (without proof here)

$\Rightarrow$ Every problem $D' \in NP$ can be reduced to SAT.
$\Rightarrow$ If we can show for some problem D that SAT can be reduced to D
then every problem $D' \in NP$ can be reduced to D.
($\preceq_p$ is transitive)

$$\forall D' \in NP \colon D' \preceq_p \text{SAT and SAT} \preceq_p D \implies \forall D' \in NP \colon D' \preceq_p D$$

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. . . and now examples !

WHITEBOARD:

SAT $\preceq_p$ 3-SAT $\preceq_p$ CLIQUE $\preceq_p$ VERTEX-COVER

3-SAT $\preceq_p$ VERTEX-COLORING

3-SAT $\preceq_p$ HAMILTONIAN PATH/CYCLE $\preceq_p$ TSP