Algorithms and Complexity

5. Greedy Algorithms

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Greedy Algorithms

A greedy algorithm always makes the choice that looks best at the moment. That is, it makes a locally optimal choice in the hope that this choice will lead to a globally optimal solution.

Example: WHITEBOARD

Minimum Spanning Tree

Let $G = (V, E)$ be a weighted, connected, undirected graph and $w(\{u, v\})$ be the weight of edge $\{u, v\}$.
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MST problem

Find a minimum spanning tree for a given weighted, connected, undirected graph.

Kruskal's Algorithm

KRUSKAL($G = (V,E), w\colon E \to \mathbb{R}$) // $m = |E|$

sort edges such that $w(e_1) \leq w(w_2) \cdots \leq w(e_m)$
$F = \emptyset$, $T = (V,F)$
for $i = 1,\ldots,m$ **do**
    if $(V, F \cup \{e_i\})$ is acyclic **then**
        $T = (V, F \cup \{e_i\})$
return T

Theorem 5.1

KRUSKAL *correctly computes an MST for a given undirected, connected graph* $G = (V,E)$ *in* $O(|E||V|)$ *time*

proof: WHITEBOARD

Matroid

A matroid is a tuple $(R, \mathbb{F})$ such that

- M1 $\mathbb{F} \neq \emptyset$ is a collection of subsets of the set R, i.e., $\mathbb{F} \subseteq \mathbb{P}(R)$.
  (Elements in $\mathbb{F}$ are called independent)
- M2 *Closed w.r.t. Inclusion:* $Y \in \mathbb{F}, X \subseteq Y \Rightarrow X \in \mathbb{F}$
- M3 *Exchange Property:* For all $X, Y \in \mathbb{F}$ and $|Y| > |X| \Rightarrow$ exists $y \in Y \setminus X$ such that $X \cup \{y\} \in \mathbb{F}$.

If $(R, \mathbb{F})$ satisfies (M1) and (M2) but not necessarily (M3), then $(R, \mathbb{F})$ is called independent system.

Many optimization problems can be formulated as independent system, where R is ground set of elements that can be chosen (eg. edges in the MST-problem) and $\mathbb{F}$ is a set of subsets of feasible solutions (eg. all spanning forests in a graph).

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Lemma 5.5

If $(R, \mathbb{F})$ is an independent system, then the following conditions are equivalent:

- M3 *For all $X, Y \in \mathbb{F}$ and $|Y| > |X| \Rightarrow$ exists $y \in Y \setminus X$ such that $X \cup \{y\} \in \mathbb{F}$.*
- M3' *For all $X, Y \in \mathbb{F}$ and $|Y| = |X| + 1 \Rightarrow$ exists $y \in Y \setminus X$ such that $X \cup \{y\} \in \mathbb{F}$.*

Proof.

chalkboard. □

Bases of an independent system $(R, \mathbb{F})$ are all maximal elements of $\mathbb{F}$.

Theorem 5.4

The basis elements of a matroid have always the same size.

Proof.

Let X, Y be bases of $\mathbb{F}$ such that $|Y| > |X|$

$\stackrel{(M3)}{\Rightarrow} \exists y \in Y \setminus X$ such that $X \cup \{y\} \in \mathbb{F}$

$\Rightarrow X$ is not maximal and thus no basis; a contradiction $\square$

GREEDY($(R, \mathbb{F}), w : R \to \mathbb{R}_{\geq 0}$) // For max-problems, min-prob. similar

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sort elements in R such that w(e_1) ≥ w(e_2) ≥ ··· ≥ w(e_m)
F = ∅
for i = 1..m do
    if F ∪ {e_i} ∈ 𝔽 then
        F = F ∪ {e_i}
return F
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Runtime: If $f(m)$ denotes the runtime to check if $F \cup \{e_i\} \in \mathbb{F}$, we have total-runtime $O(m \log(m) + mf(m))$.

Theorem 5.6

Let $(R, \mathbb{F})$ be an independent system. Then, $(R, \mathbb{F})$ is a matroid if and only if GREEDY *returns a maximum-weighted element in $\mathbb{F}$ for all weighting functions $w : R \to \mathbb{R}_{\geq 0}$.*

proof: WHITEBOARD