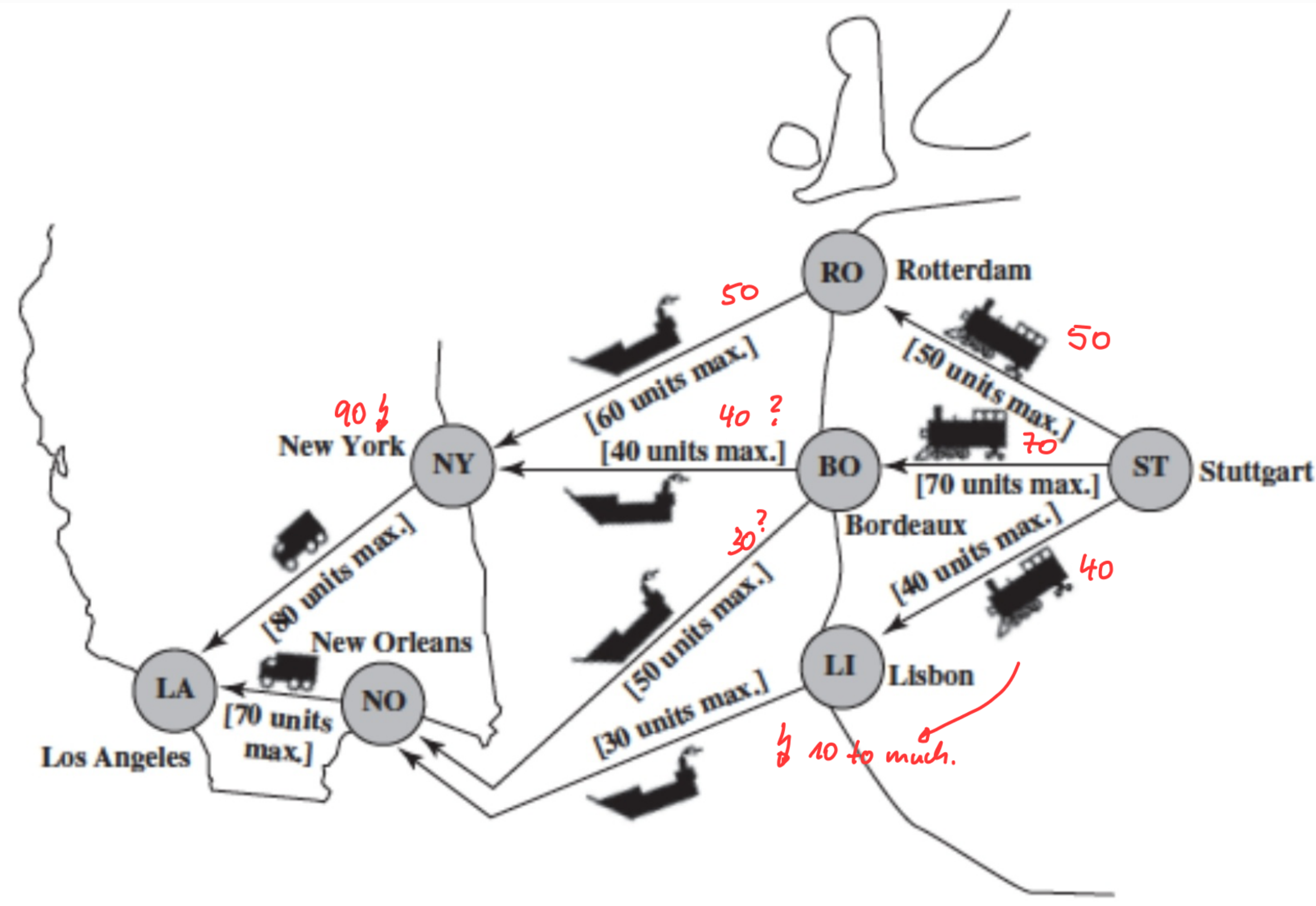
Maximum Flow



source : ST
target : LA

, daily shipment → want to maximize amount [aka: maximize flow] that is produced in ST & can be shipped on a "daily" basis.

[value] in brackets = possible capacity

if 50 + 70 + 40 (first idea)

⇒ 40 unit from ST ~~to~~ LI

but from LI only 30 can be shipped ⇒ loose 10 units !

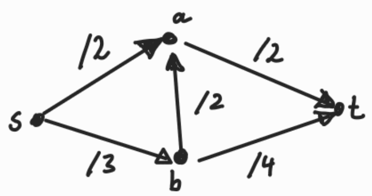
(assuming there is no extra warehouse)

Def: **Flow network** is di-graph $G=(V,E)$ with

st:
- $(v,v) \notin E \;\; \forall v \in V$
- $(uv) \in E \Rightarrow (vu) \notin E$
- unique source s // (not root & sink!)
- unique target t
- $\forall v \in V \;\exists$ path $s \rightsquigarrow v \rightsquigarrow t$ from s to t containing v

capacity function for di-graph $G=(V,E)$ is map $c: V \times V \to \mathbb{R}$ st

- $\forall u,v \in V$: $c(uv) \geq 0$ &
- $(uv) \notin E \Rightarrow c(uv) = 0$



[t may have "out-edges (tv)
s may have "in edges (vs)]

$c(sa) = 2$ $\quad$ $c(ab) = 2$
$c(sb) = 3$ $\quad$ $c(at) = 2$
$c(bt) = 4$

Def: G di-graph with capacity function c.
Then $f: V \times V \to \mathbb{R}$ is **flow** in G

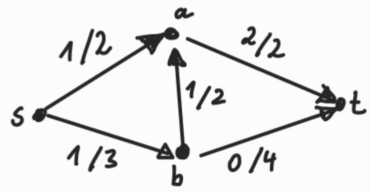
if it satisfies (F1) & (F2).

(F1) Capacity constraint:

- $0 \leq f(uv) \leq c(uv) \quad \forall u,v \in V$

(F2) Flow-conservation:

$$\sum_{v \in V} f(vu) = \sum_{v \in V} f(uv) \quad \forall u \in V \setminus \{s,t\}$$



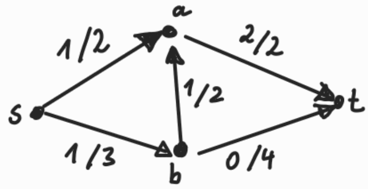
Observation 1

$$\boxed{\text{Note } (uv) \notin E \Rightarrow c(uv) = 0 \Rightarrow f(uv) = 0}$$

Def. $|f| := \sum_{v \in V} f(s,v) - \sum_{v \in V} \underbrace{f(v,s)}$

(Not absolute value or cardinality!)

usually = 0 for flow-networks since no edges (vs), BUT For later usage when considering residual netw. helpful.



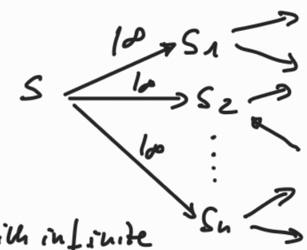
$|f| = 2$

▶ Aim is to maximize $|f|$ (=max flow f) for given flow network G & capacity c

In practice often: multiple sources or targets as well as edges $u \rightleftarrows v$.

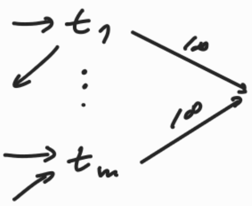
Can be resolved as follows:

$s_1 \dots s_n$ = sources: Add supersource here $|f| = \sum_{i=1}^{n} \left(\sum_{v \in V} f(s_i, v) - \sum_{v \in V} f(v, s_i) \right)$



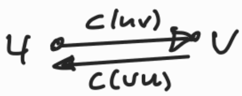
with infinite capacity $c(s, s_i)$ $\forall s_i$.

$t_1 \dots t_m$: target Add supertarget t



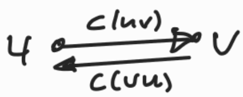
with capacity $c(t_i, t) = \infty$ $\forall t_i$

"antiparallel" edges

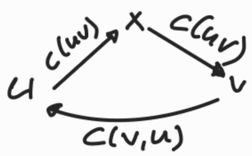


add new vertex x

and replace:



by



Exercise: show that max flow f' in modified network & max flow f in orig. network satisfy $|f| = |f'|$

method (not algorithm, since we have missing pieces & admits several ways for implementation)

FORD-FULKERSON METHOD(G, s, t, c)

1) init $f(uv) = 0 \quad \forall (u,v) \in E$
2) WHILE (exists augmenting path P in residual-network G_f)
   Augment flow f along P
3) Return f

$\Rightarrow$ Need def. "resid. network G_f"
"augmenting path"
"augmenting f along path".

Residual Network

Def: Residual capacity: $c_f(uv) := \begin{cases} c(uv) - f(uv) & ; (uv) \in E \\ f(vu) & ; (vu) \in E \\ 0 & , \text{else.} \end{cases}$

For (G, c) & flow f, the residual-network G_f
has vertex set: $V(G_f) = V(G)$
& edge set: $(uv) \in E(G_f) \Longleftrightarrow c_f(uv) > 0 \quad \forall u, v \in V(G)$

Intuition: G_f consists of all edges (uv), where we can increase flow $f(uv)$
+ „edges" to represent possibility to decrease flow, i.e. an edge to admit positive flow in opposite direction.

Example

G $\quad u \longrightarrow v$

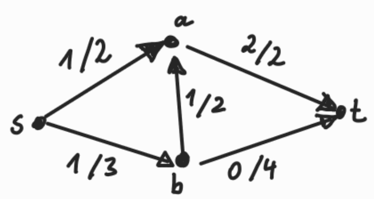
$c(uv) = 3,\ f(uv) = 2 \Rightarrow c_f(uv) = 3 - 2 = 1 > 0$
$\qquad c_f(vu) = 2 > 0$

$\rightarrow$ in G_f $\quad u \rightleftarrows v$

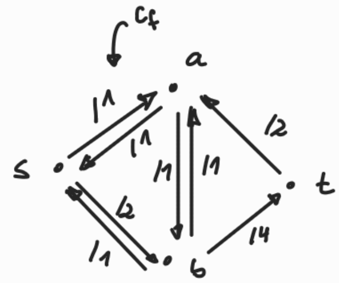
$c(uv) = 3,\ f(uv) = 3 \Rightarrow G_f \quad u \longleftarrow v$

$$\boxed{\text{Observation 2: } |E(G_f)| \leq 2|E|}$$

G:



G_f:



Flow in Res.network G_f provides a road-map for adding flow to original flow-network G

Def: If f is flow in G & f' flow in G_f we define

$f \uparrow f'$ augmentation of f by f'

as a function from $V \times V$ to $\mathbb{R}$ by

$$(f \uparrow f')(uv) := \begin{cases} f(uv) + f'(uv) - f'(vu), & \text{if } (uv) \in E(G) \\ 0, & \text{else.} \end{cases}$$

<u>Intuition:</u> "increase flow $f(uv)$ by $f'(uv)$
& decrease —"— by $f'(vu)$

<u>L. 1:</u> Given flow network $G=(V,E)$ with source s, target t, capacity c, flow f.

let f' be flow in G_f.

Then $f \uparrow f'$ is a flow in G and
$$|f \uparrow f'| = |f| + |f'|$$

<u>proof:</u> $f \uparrow f$ is flow:

<u>(F1) Capacity constraint:</u> $0 \leq f \uparrow f'(uv) \leq c(uv) \; \forall uv$

let $(uv) \notin E \Rightarrow f \uparrow f'(uv) = 0 \overset{\text{Def}}{=} c(uv)$ ✓ ⊛

let $(uv) \in E$: <u>Obs:</u> $(vu) \notin E \Rightarrow \underbrace{c_f(vu) = f(uv) \geq f'(vu)}_{\text{since } f' \text{ flow in } G_f}$ ⊛

Hence, $$f \uparrow f'(uv) \overset{\text{Def}}{=} f(uv) + f'(uv) - f'(vu)$$
$$\overset{⊛}{\geq} f(uv) + f'(uv) - f(uv)$$
$$= f'(uv) \underset{\substack{f' \text{ flow (F1)} \\ \text{in } G_f}}{\geq} 0$$

Moreover, $$f \uparrow f'(uv) \overset{\text{Def}}{=} f(uv) + f'(uv) - \underbrace{f'(vu)}_{\geq 0 \text{ (F1)}}$$
$$\leq f(uv) + f'(uv)$$
$$\underset{f' \text{ F1 in } G_f}{\leq} f(uv) + c_f(uv)$$
$$\underset{\text{Def } c_f}{=} f(uv) + c(uv) - f(uv) = c(uv) \quad ✓$$

$\Rightarrow$ capacity constraint satisfied.

$$F2: \sum_{v\in V} f\uparrow f'(uv) = \sum_{v\in V} f\uparrow f'(vu) \quad \forall\ u\in V\setminus\{s,t\}$$

$$\sum_{v\in V} f\uparrow f'(uv) \overset{\text{Def}}{=} \sum_{v\in V}\left(f(uv)+f'(uv)-f'(vu)\right)$$

$$= \sum_{v\in V} f(uv) + \sum_{v\in V} f'(uv) - \sum_{v\in V} f'(vu)$$

F2 for G & G_f

$$= \sum_{v\in V} f(vu) + \sum_{v\in V} f'(vu) - \sum_{v\in V} f'(uv)$$

$$\overset{\text{Def}}{=} \sum_{v\in V} f\uparrow f'(vu)$$

$\Rightarrow$ $f\uparrow f'$ codist. (F2).

$$|f\uparrow f| = |f| + |f'|$$

Note G has $\underline{\underline{\text{no}}}$ $u \rightleftarrows v$ edges, but G_f may have.

$\Rightarrow$ def $V_1, V_2 \subseteq V$ with $V_1 = \{v \mid (sv)\in E\}$
$V_2 = \{v \mid (vs)\in E\}$

it holds: $V_1\cup V_2 \subseteq V$, $V_1\cap V_2 = \emptyset$.

$$|f\uparrow f'| \overset{\text{Def}}{=} \sum_{v\in V}(f\uparrow f')(sv) - \sum_{v\in V}(f\uparrow f')(vs)$$

⊗ $(f\uparrow f')(vs)=0$ $\forall\ v\in V\setminus V_1$
& $(f\uparrow f')(sv)=0$ $\forall\ v\in V\setminus V_2$

$$= \sum_{v\in V_1}(f\uparrow f')(sv) - \sum_{v\in V_2}(f\uparrow f')(vs)$$

$$\underset{\text{of } f\uparrow f'}{\overset{\text{Def}}{=}} \sum_{v\in V_1} f(sv)+f'(sv)-f'(vs) - \Big(\sum_{v\in V_2} f(vs)+f'(vs)-f'(sv)\Big)$$

$$\underset{\text{terms}}{\overset{\text{reorder}}{=}} \sum_{v\in V_1} f(sv) - \sum_{v\in V_2} f(vs) + \sum_{v\in V_1\cup V_2} f'(sv) - \sum_{v\in V_1\cup V_2} f'(vs)$$

if $v\notin V_1\cup V_2$
$\Rightarrow$ neither $(sv)\in E$
nor $(vs)\in E$
$\overset{\text{Obs 1}}{\Rightarrow}$ $f(sv)=f(vs)=0$ in G

in G_f: Def $c_f(sv)=c_f(vs)=0$
$\Rightarrow$ no $(vs),(sv)$ edge in G_f
$\overset{\text{Obs!}}{\Rightarrow}$ $f'(sv)=f'(vs)=0$

$$= \underbrace{\sum_{v\in V} f(sv) - \sum_{v\in V} f(vs)}_{|f|} + \underbrace{\sum_{v\in V} f'(sv) - \sum_{v\in V} f'(vs)}_{|f'|}$$

$$= |f| + |f'|$$

$\square$

Augmenting Paths

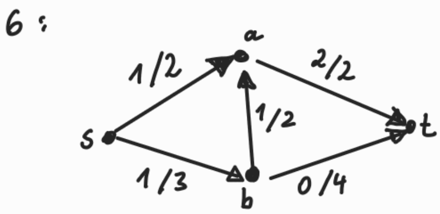
Def: For given flow-network $G=(V,E)$, cap. c, flow f an augmenting path is simple s-t path in G_f.

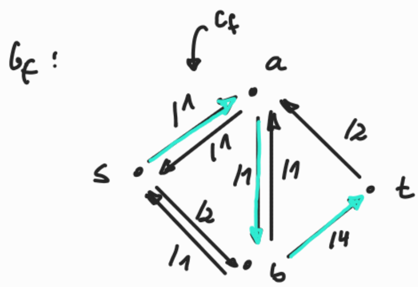
Example

Def: Residual capacity of augmenting path P is

$$c_f(P) = \min\{c_f(uv) : (uv) \text{ is edge on } P\}$$

Observation 3: edges (uv) in $G_f \overset{\text{Def}}{\Rightarrow} c_f(uv) > 0 \Rightarrow c_f(P) > 0$





increase $f(sa)$ by 1 means must decrease $f(ba)$ by 1

$s \xrightarrow{1/1} a \xrightarrow{1/1} b \xrightarrow{1/4} t$

gives possibilities: increase by 1, decrease by 1, increase by 4

Observation 4: max. value for which we can increase flow along P in G_f is $c_f(P)$

This implies next lemma.

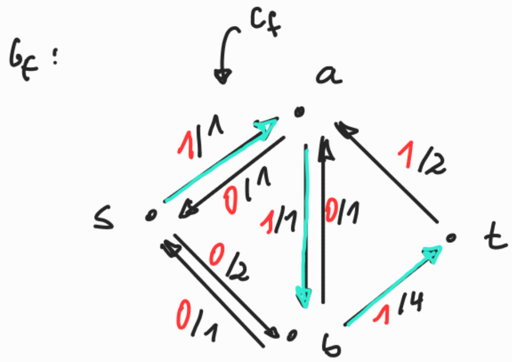
L.2: $G=(V,E)$ flow network, capacity c, flow f in G & P augmenting path in G_f.

Let $f_P : V \times V \to \mathbb{R}$ def. by

$$f_P(uv) = \begin{cases} c_f(P) &, \text{if } (uv) \text{ on } P \\ 0 &, \text{else} \end{cases}$$

OGS3.

Then f_P is flow in G_f with value $|f_P| = c_f(P) > 0$

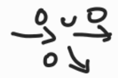


$c_f(P) = 1$

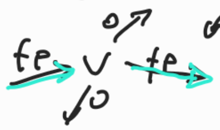
(F1): $0 \leq f_P(uv) \leq c_f(P) \leq f(uv)$ $\forall uv$ auf P

$0 = f_P(uv) \leq c_f(uv)$ else.

(F2): $v \notin P \Rightarrow$



$v \in P \Rightarrow$

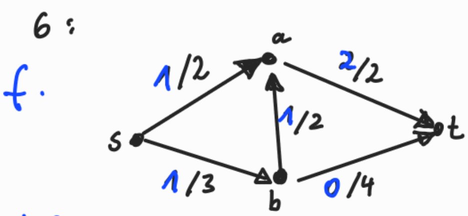


"simple path"

$\Rightarrow f_P$ Flow.

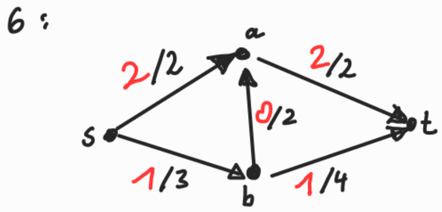
COR 1: $G=(V,E)$ flow network, capacity c, flow f in G & P augmenting path in G_f.

Then $|f \uparrow f_P| \underset{L1}{=} |f| + |f_P| \underset{L2}{>} |f|$



$|f| = 2$

$f \uparrow f_P$



$|f \uparrow f_P| = 3$

Expl $(f \uparrow f_P)(sa) = f(sa) + f_P(sa)$
$- f_P(as)$
$= 1 + 1 - 0 = 2$

$(f \uparrow f_P)(ba) = f(ba) + f_P(ba)$
$- f_P(ab)$
$= 1 + 0 - 1 = 0$

To recall:

FORD-FULKERSON METHOD(G, s, t, c)

1) init $f(uv) = 0 \quad \forall\, u, v \in V$
2) WHILE (exists augmenting path P in residual-network G_f)
    Augment flow f along P
3) Return f

Does it terminate?
And if it terminates, do we get max flow $|f|$?

We show in following that if Ford-Fulkerson method terminates, then $|f|$ is max $\Leftrightarrow$ no augm. path in G_f.

To this end: CUTS.

Def: A cut (S,T) in digraph $G=(V,E)$ is a partition of V $\quad (V = S \cup T,\ S \cap T = \emptyset)$ s.t. $s \in S$ & $t \in T$

Def: IF f flow in flownetwork G, THEN net flow $f(S,T)$ wrt cut (S,T) is

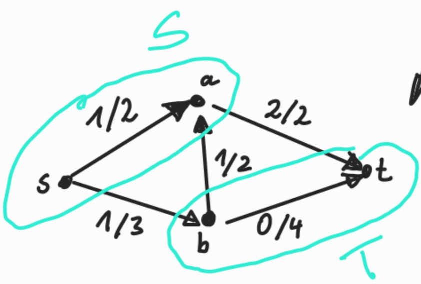
$$f(S,T) := \sum_{u \in S} \sum_{v \in T} f(uv) - \sum_{u \in S} \sum_{v \in T} f(vu)$$

and the capacity of (S,T) is

$$c(S,T) := \sum_{u \in S} \sum_{v \in T} c(uv)$$

A minimum cut is cut (S,T) that minimizes $c(S,T)$

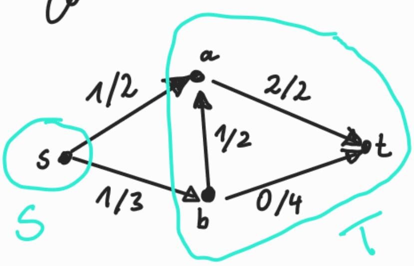
$c(uv) = 0$ & $f(uv) = 0$ by Obs 1.
$\forall\, u,v \in V$ with $(uv) \notin E$



$$f(S,T) = f(sb) + f(at) - f(ba)$$
$$= 1 + 2 - 1 = 2 = |f|$$

$$c(S,T) = c(sb) + c(at)$$
$$= 3 + 2 = 5$$

another cut!



$$f(S,T) = f(sa) + f(sb) = 1 + 1 = |f|$$

L3: let f be flow in flow-network G, source s, target t, capacity c
& (S,T) be any cut in G.
THEN
$$f(S,T) = |f|$$

proof:

$$(F2): \sum_{v\in V} f(uv) - \sum_{v\in V} f(vu) = 0 \quad \forall u \in V\setminus\{s,t\} \quad (\ast\ast)$$

this holds in particular for $u\in S\setminus\{s\}$ since $t\notin S$

$$|f| \overset{\text{Def}}{=} \sum_{v\in V} f(sv) - \sum_{v\in V} f(vs) + 0$$

$$\overset{(\ast\ast)}{=} \sum_{v\in V} f(sv) - \sum_{v\in V} f(vs) + \sum_{u\in S\setminus\{s\}}\Big(\sum_{v\in V} f(uv) - \sum_{v\in V} f(vu)\Big)$$

regroup

$$= \sum_{v\in V}\Big(f(sv) + \sum_{u\in S\setminus\{s\}} f(uv)\Big) - \sum_{v\in V}\Big(f(vs) + \sum_{u\in S\setminus\{s\}} f(vu)\Big)$$

$$= \sum_{v\in V}\sum_{u\in S} f(uv) - \sum_{v\in V}\sum_{u\in S} f(vu)$$

$V = S \,\dot\cup\, T$

$$= \underbrace{\cancel{\sum_{v\in S}\sum_{u\in S} f(uv)}}_{x\in S,\, y\in S \to f(xy)} + \sum_{v\in T}\sum_{u\in S} f(uv) - \underbrace{\cancel{\sum_{v\in S}\sum_{u\in S} f(vu)}}_{y\in S\; x\in S\; f(xy)} - \sum_{v\in T}\sum_{u\in S} f(vu)$$

i.e. the 2 sums are the same & cancel out.

$$= \sum_{u\in S}\sum_{v\in T} f(uv) - \sum_{u\in S}\sum_{v\in T} f(vu) \overset{\text{Def}}{=} f(S,T)$$

Since (S,T) was chosen arbitrarily, the statement follows. $\square$

COR2: $|f| \leq c(S,T)$ for any cut (S,T) in flow-network G with flow f.

proof:

$$|f| \underset{L3.}{=} f(S,T) \underset{Def}{=} \sum_{u \in S} \sum_{v \in T} f(uv) - \sum_{u \in S} \sum_{v \in T} f(vu)$$

$$\leq \sum_{u \in S} \sum_{v \in T} (f(uv))$$

$$\underset{(F1)}{\leq} \sum_{u \in S} \sum_{v \in T} c(uv) \underset{Def}{=} c(S,T)$$

$\square$

Next theorem show that $|f| = c(S,T)$ in case f max flow.

Theorem 1 [MAX-FLOW MIN-CUT THM]

IF f flow in flow network G with source s, target t, capacity c

THEN the following statements are equivalent:

(1) f is max. flow in $G = (V,E)$
(2) G_f has no augmenting path
(3) $|f| = c(S,T)$ for some cut (S,T) of G.

By COR 2: $|f| \leq c(S,T)$ $\forall$ cuts (S,T)

$\Rightarrow$ if $|f| = c(S,T)$, then (S,T) is minimum cut proof:

$(1 \Rightarrow 2)$: Let f max flow in G.

IF (for contradiction) G_f has augm. path P

$\overset{\text{COR 1}}{\Longrightarrow}$ $f \uparrow f_P$ flow in G & $|f \uparrow f_P| > 0$ ↯ to f max

$(2 \Rightarrow 3)$: Suppose G_f has no augmenting paths.

Let $S := \{v : \exists \text{ path from } s \text{ to } v \text{ in } G_f\}$

& $T := V \setminus S$

clearly (S,T) is cut: since $s \in S$ (by def)

& since no s-t path

$\Rightarrow t \notin S \Rightarrow t \in T$.

Let $u \in S$ & $v \in T$

IF $(uv) \in E(G) \Rightarrow f(uv) = c(uv)$ since G_f only edges $(uv) \in E(G)$ for which $c_f(uv) := c(uv) - f(uv) > 0$ but $\nexists$ uv-path in G_f.

IF $\underline{(vu) \in E(G)} \Rightarrow \underline{f(v,u) = 0}$ since then $c_f(uv) = f(vu)$ & $(uv) \notin E(G)$ is edge in G_f iff $c_f(uv) > 0$ but $\nexists$ uv-path in G_f.

IF $(uv) \notin E(G)$, $\underline{(vu) \notin E(G)}$ $\Rightarrow f(uv) = \underline{f(vu) = 0}$ by def.

$$\Rightarrow f(S,T) \overset{\text{Def}}{=} \sum_{u \in S} \sum_{v \in T} f(uv) - \underbrace{\sum_{u \in S} \sum_{v \in T} f(vu)}_{=0}$$

$$\underbrace{\sum_{\substack{u \in S\\ v \in T\\ uv \notin E(G)}} f(uv)}_{=0} + \underbrace{\sum_{\substack{u \in S\\ v \in T\\ uv \in E(G)}} f(uv)}_{= \sum c(uv)}$$

$$= \sum_{u \in S} \sum_{v \in T} c(uv) \overset{\text{Def}}{=} c(S,T)$$

$$\overset{\text{L.3}}{\Longrightarrow} |f| = f(S,T) = c(S,T).$$

$(3 \Rightarrow 1)$: By Cor. 2 $|f| \leq c(S,T)$ $\forall$ cut (S,T) of G.
$\Rightarrow$ if $|f| = c(S,T)$ then f max flow in G.

$\checkmark$ $\square$

To recall again:

FORD-FULKERSON METHOD(G,s,t,c)

1) init $f(uv) = 0$ $\forall$ $u,v \in V$
2) WHILE (exists augmenting path P in residual-network G_f)
   Augment flow f along P $\hat{=}$ Replace f by $f \uparrow f_P$
3) Return f

By the latter arguments, in each step (WHILE) we get f_P & $f \uparrow f_P$ st $|f \uparrow f_P| > |f|$.

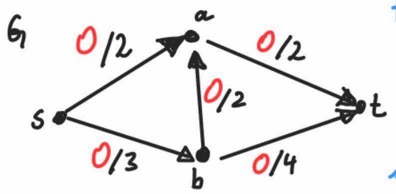
$$c_f(uv) := \begin{cases} c(uv) - f(uv) & ; (uv) \in E \\ f(vu) & ; (vu) \in E \\ 0 & , \text{ else.} \end{cases}$$

Refined version:

FORD-FULKERSON(G,c,s,t)

1. init $f(uv) = 0$ $\forall$ $(uv) \in E$
2. WHILE (exists st-path P in G_f)
   1. $c_f(P) = \min\{c_f(uv) : (uv) \text{ edge in } P\}$
   2. FOR (all edges (uv) in P)
   3. IF $((uv) \in E(G))$
   4. $f(uv) \leftarrow f(uv) + c_f(P)$
   5. ELSE $f(vu) \leftarrow f(vu) - c_f(P)$
3. RETURN f

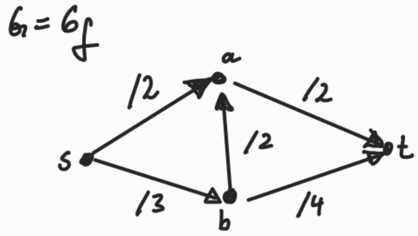
After step 1



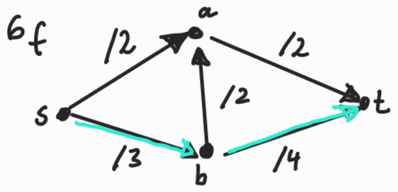
for all non-edges (uv) in G: $c(uv)=0$, $f(uv)=0$

$\Rightarrow c_f(uv)=0$

for all edges (uv) in G: $c_f(uv)>0$

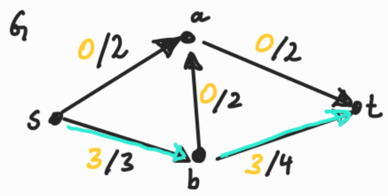


step 2:

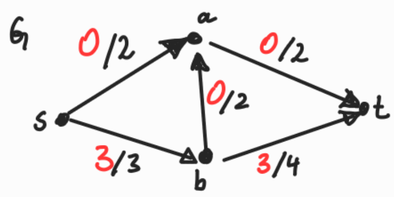


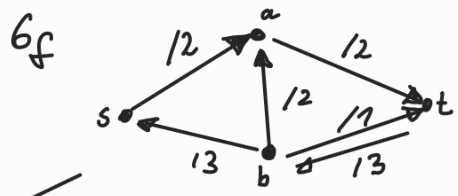
st-path P $\Rightarrow f_P$!

$c_f(P)=3$

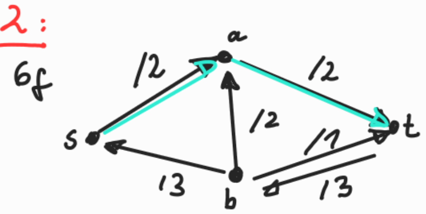


$\Rightarrow$ in G

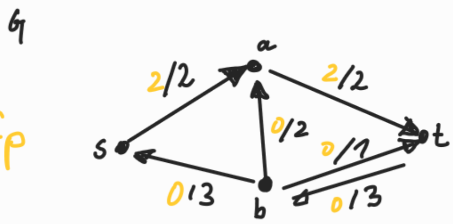




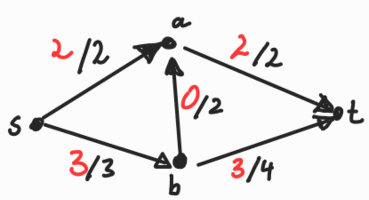
step 2:

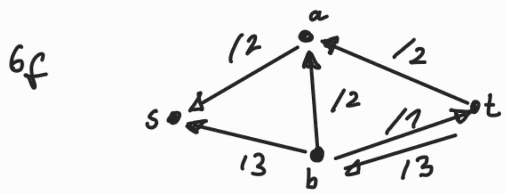


$c_f(P)=2 \rightarrow f_P$



$\Rightarrow$ in G





no augmenting path $\Rightarrow$ stops

& f in G with max value $|f|=5$ returned.

Analysis of FORD-FULKERSON:

in each step of WHILE, it is ensured by COR 3 that $f \leftarrow f \uparrow f_P$ strictly increases.
& if terminates $\Rightarrow$ we get max flow f of G.

Problem: it may not converge to max. flow.
e.g. if capacities are irrational numbers

nevertheless if integer-capacities are used, the Alg clearly terminates, since f increased by at least 1 in each step

suppose capacities are integer. (if real-number $\rightarrow$ rescale to integers)
If f^* denotes max flow in G.

$\Rightarrow$ Step 1. $O(|E|)$

Step 2. WHILE loop at most f^*-times.
where "finding s-t path" goes in $O(|V|+|E_f|) \overset{Obs.2}{=} O(|V|+|E|) = O(E)$ (G conn.) time.
via BFS.

2.1. $c_f(P)$ can be tracked when finding path [constant]
2.2. $O(|E|)$ since $O(E)=O(|E_f|)$
2.3–2.5 constant.

$\Rightarrow$ $O(|f^*||E|)$ $\leftarrow$ with smart maintainance of f goes in $O(|f^*||E|)$ time.

[$|f^*| = 10^6 \Rightarrow$ then problem!]

IF however in WHILE we choose shortest st-path in G_f it is guaranteed, that Alg. always terminates.
$\nearrow$ this version know as EDMONDS-KARP alg.
($O(|V||E|^2)$ time.
[omitted $\nearrow$ Book Introd. to Alg Chp 26).

Simple final FUN-example:

[+recheck max matching in bip. graphs].

3 room-mates Ann, John, Kate.

expences: John 40 SEK
Ann 10 SEK
Kate 10 SEK.

how to split the expences equally:

clearly: $40+10+10 = 60$ expences total

$\Rightarrow$ each has to pay 20 SEK.

$\Rightarrow$ Ann/Kate only paid 10 $\Rightarrow$ still have to pay 10

$$\text{Diff}(\text{Ann}) = -10$$
$$\text{Diff}(\text{Kate}) = -10$$
$$\text{Diff}(\text{John}) = 20$$

Here easy

$\Rightarrow$ makes 20 in total still to pay to John

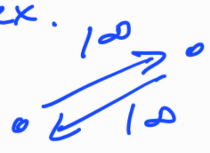
& Ann + Kate gives each 10 to John.

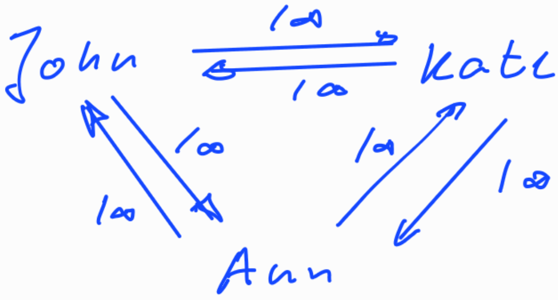
gets easily complicated if more people are involved.

Altern. solution: Max Flow.

But $e = \sum$ paid - expenses

For each person: add vertex.

& connect all persons via 

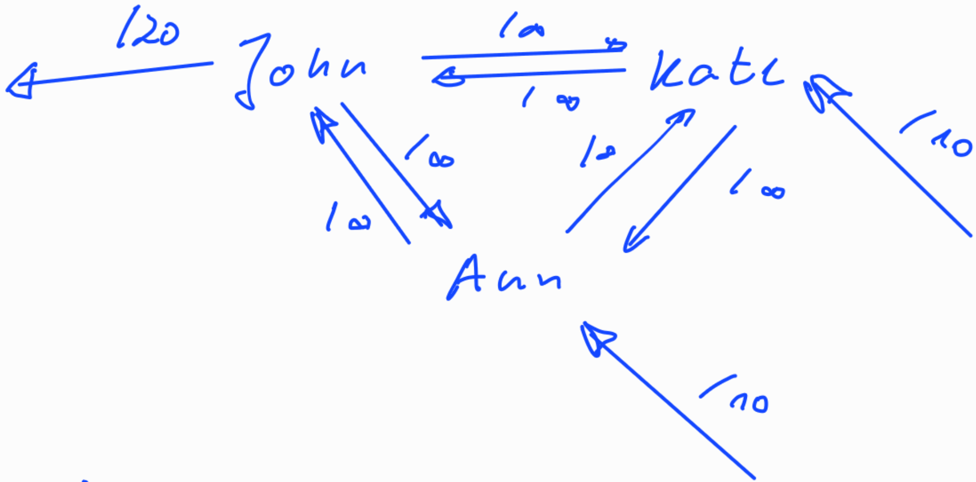


+ add to each person v:

edge (v, t_v) if $\mathrm{Diff}(v) \geq 0$

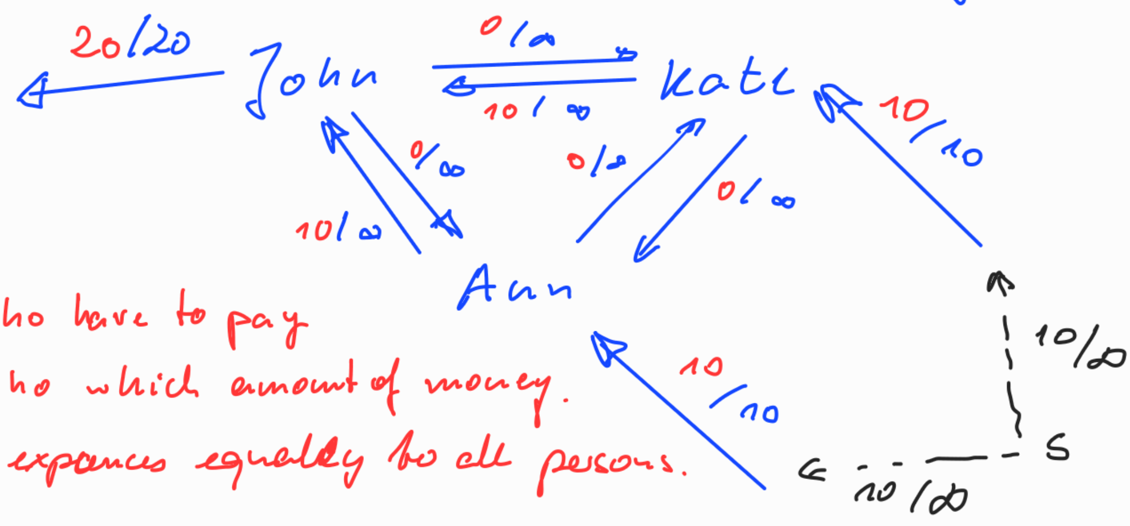
(s_v, v) if $\mathrm{Diff}(v) \leq 0$

with capacity $|\mathrm{Diff}|$



may have several sources s_v targets t_v ⇝ adj.

<u>max flow-Alg</u>



specifies who have to pay who which amount of money.

⇒ dividing expences equally to all persons.