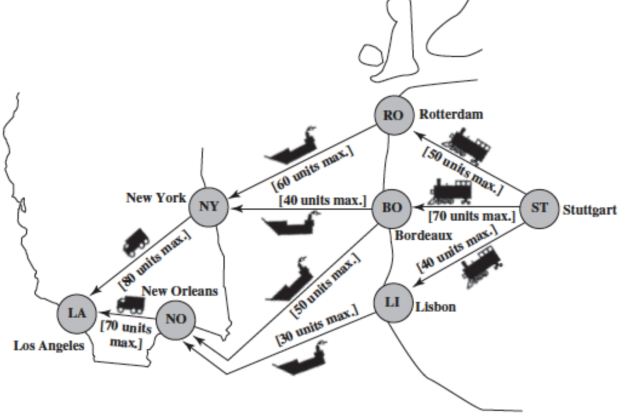
Algorithms and Complexity

Maximum Flow and Flow Networks

Marc Hellmuth

University of Stockholm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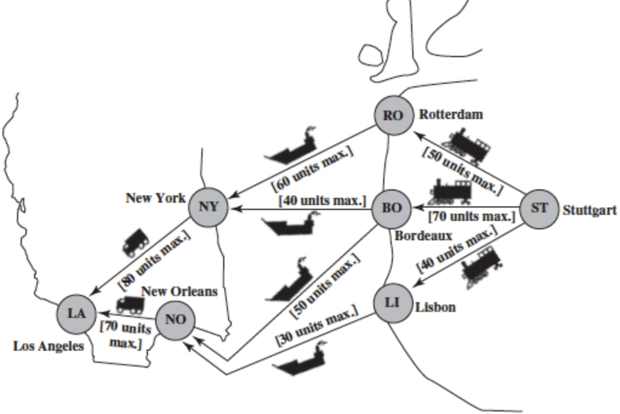
Los Angeles



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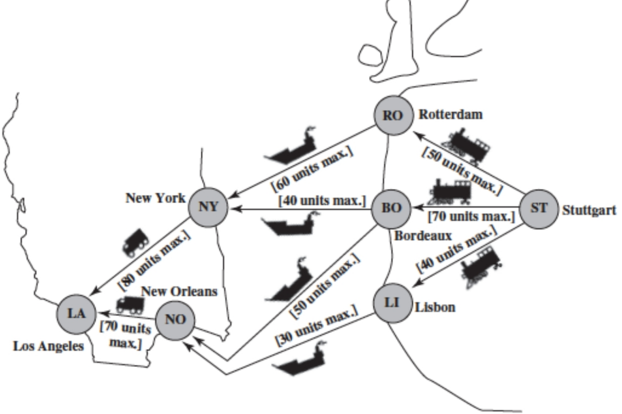

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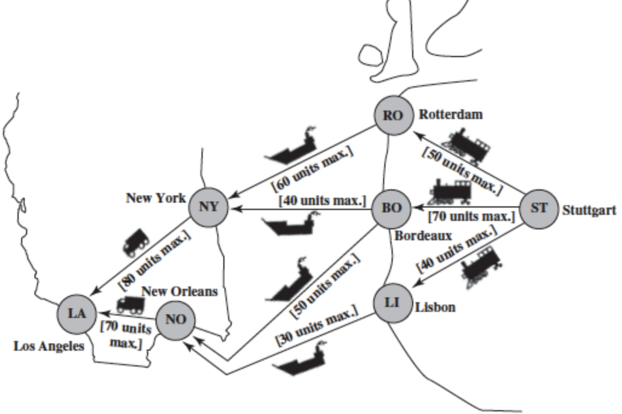

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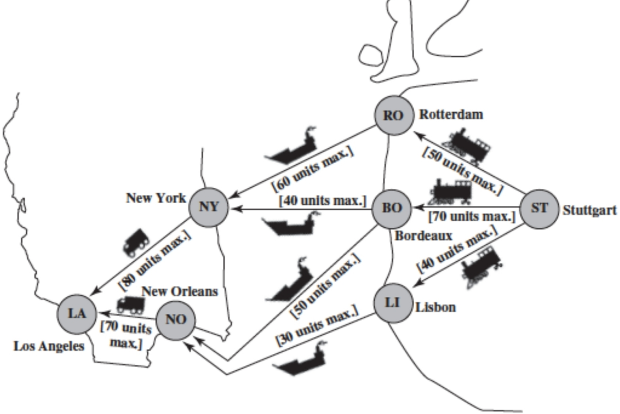

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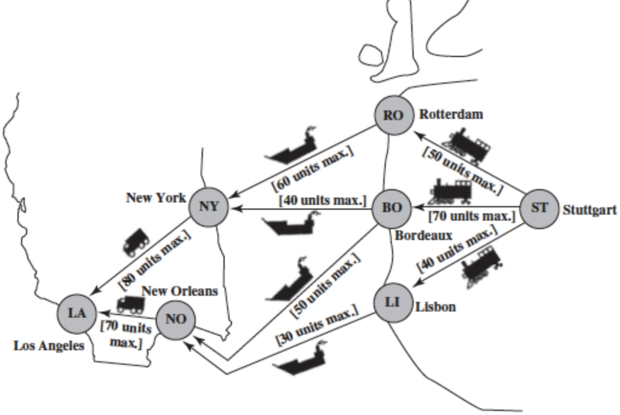

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- the system is in a steady state, i.e. the amount of material leaving a vertex is equal to the amount of material entering the vertex [= no possibility to store material]
- all material ends at a vertex called target, where it is consumed

Flow Network and Maximum Flow

A flow network $G = (V, E)$ is a directed graph such that

- G has no self-loops
- $(u, v) \in E \Rightarrow (v, u) \notin E \qquad (\forall u, v \in V)$
- V has two distinct vertices, a source s and a target t
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For a di-graph $G = (V, E)$ a capacity funktion is a map $c\colon V \times V \to \mathbb{R}$ such that

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For a di-graph $G = (V,E)$ with cap.fct. c, a flow in G is a function $f : V \times V \to \mathbb{R}$ such that

F1 capacity constraint: $0 \leq f(u,v) \leq c(u,v) \qquad (\forall u,v \in V)$
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In practice, we often have multiple sourses, targets or anti-parallel edges – no problem!
[WHITEBOARD]

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$$c_f(u,v) = \begin{cases} c(u,v) - f(u,v) & \text{, if } (u,v) \in E \\ f(v,u) & \text{, if } (v,u) \in E \\ 0 & \text{, otherwise.} \end{cases}$$

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Let f be a flow in G and f' be a flow in the residual network G_f. Define $f \uparrow f' : V \times V \to \mathbb{R}$ – the augmentation of f by f' – as follows

$$(f \uparrow f')(u,v) = \begin{cases} f(u,v) + f'(u,v) - f'(v,u) & \text{, if } (u,v) \in E \\ 0 & \text{, otherwise.} \end{cases}$$

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Lemma 1: Given flow network $G = (V,E)$ with source s, target t, capacity fct. c and flow f and f' be flow in G_f. Then, $f \uparrow f'$ is a flow in G and satisfies $|f \uparrow f'| = |f| + |f'|$.

[Proof - WHITEBOARD]

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Given flow network $G = (V, E)$ with source s, target t and capacity fct. c and flow f.

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- The flow $f_p \colon V \times V \to \mathbb{R}$ defined by augmenting path P is

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[Proof - WHITEBOARD]

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Cor. 1: Given flow network $G = (V, E)$ with source s, target t, capacity fct. c, flow f and P be an augmenting path in G_f. Then, $|f \uparrow f_P| = |f| + |f_P| > |f|$.

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Show, that if it terminates, then we a mximum flow f of G is returned.
To this end, we need the definition of CUTS!

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- The net flow $f(S, T)$ across the cut (S, T) is

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$$f(S,T)=\sum_{u\in S}\sum_{v\in T}f(u,v)-\sum_{u\in S}\sum_{v\in T}f(v,u).$$

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Thm [Max-Flow Min-Cut Theorem] Given flow network G with source s, target t, capacity fct. c and flow f. The following statements are quivalent:

1) f is a maximum flow in G.
2) The residual network G_f contains no augmenting paths.
3) $|f| = c(S, T)$ for some cut (S, T) of G.

[Proof - WHITEBOARD]

FordFulkerson_Algorithm(G, s, t, c)

```
for each edge (u,v) ∈ E do f(u,v) ← 0
while there is a path P from s to t in the residual network G_f do
    c_f(P) ← min{c_f(u,v) | (u,v) is on P}
    for each edge (u,v) in p do
        if (u,v) ∈ E then
            f(u,v) ← f(u,v) + c_f(P)
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By the latter arguments, it is ensured, that if this alg. terminates, we get maximum flow.

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Runtime (assuming that f^* is returned max flow):

- Line 1: $O(|E|)$ (via BFS in $O(|V|+|E|)$ time and since G "connected" and so $|E| \geq |V|-1$")
- Line 2: is called at most $|f^*|$ times and finding st-path can be done in $O(|E|)$ time.
- Line 3: When constructing P in Line 2, we can keep track of $c_f(P)$, i.e., constant time in Line 3
- Line 4: $O(|E|)$ calls (with Line 5,6 constant time)

In total $O(|f^*||E|)$ time - which can cause problem if e.g. $|f^*| = 10^9$.

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If one chooses ”shortest st-paths” in Line 2 we obtain the EDMONDS-KARP algorithm which has runtime $O(|V||E|^2)$ time [Detail Chp 26 in Course Book]