Section 10.5

1. (a) $\int \underbrace{x}_{f}\underbrace{e^{-x}}_{g'}\,dx = \underbrace{x}_{f}\underbrace{(-e^{-x})}_{g} - \int \underbrace{1}_{f'}\underbrace{(-e^{-x})}_{g}\,dx = -xe^{-x} + (-e^{-x}) + c = -e^{-x}(x+1) + c$

(b) $\int \underbrace{3x}_{f}\underbrace{e^{4x}}_{g'}\,dx = 3x\cdot\frac{1}{4}e^{4x} - \int 3\cdot 1\cdot \frac{1}{4}e^{4x}\,dx = \frac{3}{4}xe^{4x} - \frac{3}{4}\cdot\frac{1}{4}e^{4x} + c = \frac{3}{16}e^{4x}(4x-1)+c$

(c) $\int \underbrace{(1+x^2)}_{f}\underbrace{e^{-x}}_{g'}\,dx = (1+x^2)(-e^{-x}) - \int 2x(-e^{-x})\,dx = -(1+x^2)e^{-x} + 2\int \underbrace{x}_{f}\underbrace{e^{-x}}_{g'}\,dx =$

$= -(1+x^2)e^{-x} + 2\left(-xe^{-x} + \int e^{-x}dx\right) = -(1+x^2)e^{-x} - 2xe^{-x} - 2e^{-x} + c = -e^{-x}(x^2+2x+3)+c$

(d) $\int \underbrace{x}_{g'}\underbrace{\ln x}_{f}\,dx = \frac{x^2}{2}\ln x - \int \frac{x^2}{2}\cdot\frac{1}{x}dx = \frac{x^2}{2}\ln x - \int\frac{x}{2}dx = \frac{x^2}{2}\ln x - \frac{x^2}{4}+c$

2. (a) $\int_{-1}^{1} \underbrace{x}_{g'}\underbrace{\ln(x+2)}_{f}\,dx = \frac{x^2}{2}\ln(x+2)\Big|_{-1}^{1} - \int_{-1}^{1}\frac{x^2}{2}\cdot\frac{1}{x+2}dx = \frac{1}{2}\ln 3 - \frac{1}{2}\int_{-1}^{1}\frac{x^2}{(x+2)}dx =$

$$\begin{array}{r|l} x+2 & x-2 \\ \hline & x^2 \\ & x^2+2x \\ \hline & // \quad -2x \\ & -2x-4 \\ \hline & // \quad 4 \end{array}$$

$= \frac{1}{2}\ln 3 - \frac{1}{2}\int_{-1}^{1}\left(x-2+\frac{4}{x+2}\right)dx =$

$= \frac{1}{2}\ln 3 - \frac{1}{2}\left(\frac{x^2}{2} - 2x + 4\ln(x+2)\right)\Big|_{-1}^{1} =$

$= \frac{1}{2}\ln 3 - \frac{1}{2}\left(\frac{1}{2} - 2 + 4\ln 3\right) + \frac{1}{2}\left(\frac{1}{2}+2\right) = -\frac{3}{2}\ln 3 + 2$

(b) $\int_0^2 \underbrace{x}_{f}\underbrace{2^x}_{g'}\,dx = \frac{1}{\ln 2}\cdot x2^x\Big|_0^2 - \int_0^2 1\cdot 2^x\cdot\frac{1}{\ln 2}dx = \frac{1}{\ln 2}(2\cdot 2^2 - 0) - \frac{2^x}{(\ln 2)^2}\Big|_0^2 = \frac{8}{\ln 2} - \frac{4}{(\ln 2)^2} + \frac{1}{(\ln 2)^2} = \frac{8}{\ln 2} - \frac{3}{\ln 2 \ln 2}$

(c) $\int_0^1 \underbrace{x^2}_{f}\underbrace{e^x}_{g'}\,dx = x^2e^x\Big|_0^1 - \int_0^1 \underbrace{2x}_{f}\cdot\underbrace{e^x}_{g'}dx = e - 2xe^x\Big|_0^1 + \int_0^1 2e^x dx = e - 2e + 2e^x\Big|_0^1 = -e+2e-2 =$
$= e-2$

(d) $\int_0^3 \underbrace{x}_{f}\underbrace{\sqrt{1+x}}_{g'}\,dx = \frac{2}{3}x(1+x)^{3/2}\Big|_0^3 - \int_0^3 \frac{2}{3}(1+x)^{3/2}dx = \frac{2}{\cancel{3}}\cdot\cancel{3}\cdot 4^{3/2} - \frac{2}{3}\cdot\frac{2}{5}\cdot(1+x)^{5/2}\Big|_0^3 =$

$= 16 - \frac{4}{15}\left(4^{5/2}-1\right) = 16 - \frac{4}{15}(32-1) = 16 - \frac{4\cdot 31}{15}$

Section 17.1

1. $f(x,y) = -2x^2 - y^2 + 4x + 4y - 3$ $\quad f'_x(x,y) = -4x+4 \quad f'_y(x,y) = -2y+4$

$\begin{cases} -4x+4=0 \\ -2y+4=0 \end{cases} \Rightarrow \begin{cases} x=1 \\ y=2 \end{cases}$ $(1,2)$ is the only critical point so it has to be the max

4. $P(x,y) = -x^2 - y^2 + 22x + 18y - 102$ (a) $x=10, y=8 \Rightarrow P(10,8) = -100-64+220+144-102 =$
$= 98$ $\quad P(12,10) = -144-100+264+180-102 = 98$

(b) $P'_x(x,y) = -2x+22 \quad P'_y(x,y) = -2y+18 \rightarrow \begin{cases} -2x+22=0 \\ -2y+18=0 \end{cases} \Rightarrow \begin{cases} x=11 \\ y=9 \end{cases}$ $(11,9)$ gives max profit

The max profit is $P(11,9) = -121-81+242+162-102 = 100$

Section 17.2

4. $p = 25-x \quad q = 24-2y \quad C(x,y) = 3x^2+3xy+y^2$

(a) The profit is $\pi(x,y) = p\cdot x + q\cdot y - C(x,y) = (25-x)x + (24-2y)y - 3x^2 - 3xy - y^2 =$

$= 25x - x^2 + 24y - 2y^2 - 3x^2 - 3xy - y^2 = -4x^2 - 3y^2 - 3xy + 25x + 24y$

(b) $\pi'_x(x,y) = -8x - 3y + 25 \qquad \pi'_y(x,y) = -6y - 3x + 24$

$\begin{cases} -8x - 3y + 25 = 0 \\ -6y - 3x + 24 = 0 \end{cases} \Rightarrow \begin{cases} -16x - 6y + 50 = 0 \\ -3x - 6y + 24 = 0 \end{cases}$ subtract $\Rightarrow -13x + 26 = 0 \Rightarrow x = 2$

$\Rightarrow -3\cdot 2 - 6y + 24 = 0 \Rightarrow 6y = 24 - 6 \Rightarrow 6y = 18 \Rightarrow y = 3$ $\quad x = 2, y = 3$ gives maximum profit

$\pi''_{xx}(x,y) = -8 \qquad \pi''_{yy}(x,y) = -6 \qquad \pi''_{xy}(x,y) = -3$

$H = \begin{pmatrix} -8 & -3 \\ -3 & -6 \end{pmatrix} \Rightarrow \det H = 48 - 9 = 39 > 0 \Rightarrow$ since $\pi''_{xx} < 0$, $(2,3)$ is max point

5. $C(x,y) = x^2 + xy + y^2 + x + y + 14 \qquad \frac{1}{2}p + \frac{1}{2} < q < 2p - 1$

$\pi(x,y) = px + qy - C(x,y) = -x^2 - xy - y^2 + (p-1)x + (q-1)y - 14$

$\pi'_x(x,y) = -2x - y + p - 1 \qquad \pi'_y(x,y) = -x - 2y + q - 1$

$\begin{cases} -2x - y + p - 1 = 0 \\ -x - 2y + q - 1 = 0 \end{cases} \Rightarrow \begin{cases} y = -2x + p - 1 \\ -x + 4x - 2p + 2 + q - 1 = 0 \end{cases} \Rightarrow \begin{cases} y = -2x + p - 1 \\ 3x = 2p - q - 1 \end{cases} \Rightarrow \begin{cases} y = \frac{-4p + 2q + 2 + 3p - 3}{3} \\ x = \frac{2p - q - 1}{3} \end{cases}$

$\begin{cases} y = \frac{-p + 2q - 1}{3} \\ x = \frac{2p - q - 1}{3} \end{cases}$ these values are for crit. point

$\pi''_{xx} = -2 \qquad \pi''_{yy} = -2 \qquad \pi''_{xy} = -1 \Rightarrow H = \begin{pmatrix} -2 & -1 \\ -1 & -2 \end{pmatrix} \Rightarrow \det H = 4 - 1 = 3 > 0 \Rightarrow \left(\frac{2p-q-1}{3}, \frac{-p+2q-1}{3}\right)$ max

Section 17.3

2. $f(x,y) = x^2 + 2xy^2 + 2y^2 \qquad f'_x(x,y) = 2x + 2y^2 \qquad f'_y(x,y) = 4xy + 4y$

$f''_{xx}(x,y) = 2 \qquad f''_{yy}(x,y) = 4x + 4 \qquad f''_{xy}(x,y) = 4y$

$\begin{cases} 2x + 2y^2 = 0 \\ 4xy + 4y = 0 \end{cases} \Rightarrow \begin{cases} 2x + 2y^2 = 0 \\ 4y(x+1) = 0 \end{cases}$ $\begin{cases} x = 0 \\ y = 0 \end{cases}$ $\begin{cases} 2y^2 = 2 \\ x = -1 \end{cases} \Rightarrow \begin{cases} y = \pm 1 \\ x = -1 \end{cases}$ crit. points are $(0,0)$, $(-1,1)$, $(-1,-1)$

$H(x,y) = \begin{pmatrix} 2 & 4y \\ 4y & 4y + 4 \end{pmatrix} \qquad H(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix} = 8 > 0 \Rightarrow (0,0)$ minimum

$H(-1,1) = \begin{pmatrix} 2 & 4 \\ 4 & 8 \end{pmatrix} = 16 - 16 = 0$ undecided $\qquad H(-1,-1) = \begin{pmatrix} 2 & -4 \\ -4 & 0 \end{pmatrix} = -16$ saddle point

5. $f(x,y) = \ln(1 + x^2 y)$ (a) defined on $\{1 + x^2 y > 0\}$

(b) $f'_x(x,y) = \frac{2xy}{1 + x^2 y} \qquad f'_y(x,y) = \frac{x^2}{1 + x^2 y} \qquad \begin{cases} 2xy = 0 \\ x^2 = 0 \end{cases} \Rightarrow$ all point of $\{x = 0\}$ (y-axis) are critical

$f''_{xx}(x,y) = \frac{2y(1+x^2y) - 2xy \cdot 2xy}{(1+x^2y)^2} = \frac{2y + 2x^2y^2 - 4x^2y^2}{(1+x^2y)^2} = \frac{2y - 2x^2y^2}{(1+x^2y)^2}$

$f''_{yy}(x,y) = \frac{-x^2(x^2)}{(1+x^2y)^2} = -\frac{x^4}{(1+x^2y)^2} \qquad f''_{xy}(x,y) = \frac{2x(1+x^2y) - x^2 \cdot 2xy}{(1+x^2y)^2} = \frac{2x + \cancel{2x^3y} - \cancel{2x^3y}}{(1+x^2y)^2} = \frac{2x}{(1+x^2y)^2}$

$H(0,y) = \begin{pmatrix} \frac{2y}{(1+y)^2} & 0 \\ 0 & 0 \end{pmatrix} = 0$ undecided. But $f(x,y) > 0$ when $x^2 y > 0 \Rightarrow$ when $y > 0$ while $f(x,y) < 0$ when $y < 0$, so all the critical points are saddle points.