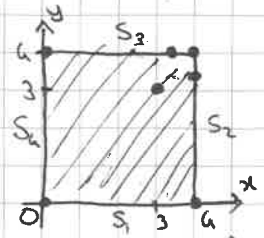
Section 17.5



2. (a) $f(x,y) = x^3+y^3-9xy+27$ $\qquad S = \{(x,y): 0 \le x \le 4,\ 0 \le y \le 4\}$

INTERIOR: $\{0<x<4,\ 0<y<4\}$ $\qquad f'_x(x,y) = 3x^2-9y$

$f'_y(x,y) = 3y^2-9x$ $\qquad \begin{cases} 3x^2-9y=0 \\ 3y^2-9x=0 \end{cases} \Rightarrow \begin{cases} y=\frac{1}{3}x^2 \\ 3\cdot\frac{1}{9}x^4-9x=0 \end{cases} \Rightarrow \begin{cases} y=\frac{x^2}{3} \\ x^4-27x=0 \end{cases} \Rightarrow \begin{cases} y=\frac{x^2}{3} \\ x=0,\ x=3 \end{cases}$

$\begin{cases} y=0 \\ x=0 \end{cases}$ $\qquad \begin{cases} y=9/3=3 \\ x=3 \end{cases}$ $\qquad (0,0)$ and $(3,3)$ are critical points

BOUNDARY: S_1 : $y=0,\ 0\le x\le 4 \Rightarrow g_1(x) = x^3+27 \Rightarrow g'_1(x) = 3x^2.\ 3x^2=0 \Leftrightarrow x=0$

the only critical point is $x=0$, we also add the endpoint $x=4$ (so $(0,0)$ and $(4,0)$)

S_2 : $x=4,\ 0\le y\le 4 \Rightarrow g_2(y) = 64+y^3-36y+27 = y^3-36y+91$

$g'_2(x,y) = 3y^2-36 \Rightarrow 3y^2-36=0 \Rightarrow y^2=12 \Rightarrow y=\pm2\sqrt{3}$ we have a crit. point

$(4,2\sqrt{3})$ and we also add the endpoint $(4,4)$.

S_3 : $y=4,\ 0\le x\le 4 \Rightarrow g_3(x) = x^3+64-36x+27 = x^3-36x+91 \Rightarrow g'_3(x) = 3x^2-36$

$3x^2-36=0 \Leftrightarrow x=\pm2\sqrt{3}$ we have a crit. point $(2\sqrt{3},4)$ and we add the endpoint $(0,4)$

S_4: $x=0,\ 0\le y\le 4 \Rightarrow g_4(y) = y^3+27$ $\quad g'_4(y)=3y^2 \Rightarrow$ only crit. point is $y=0$, but we have $(0,0)$ already.

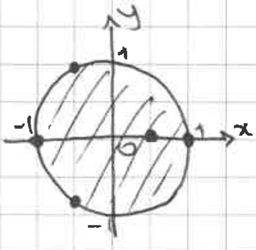
Candidates: $(0,0), (4,0), (0,4), (4,4), (2\sqrt{3},4), (4,2\sqrt{3}), (3,3)$

$f(0,0)=27$ $\quad f(4,0)=91$ $\quad f(0,4)=91$ $\quad f(4,4)=11$ $\quad f(2\sqrt{3},4) = 91-48\sqrt{3} \approx 7.86$

$f(4,2\sqrt{3}) = 91-48\sqrt{3}$ $\quad f(3,3)=0$

MIN at $(3,3)$ $\qquad$ MAX at $(4,0)$ and $(0,4)$

(b) $f(x,y) = x^2+2y^2-x$ $\qquad S=\{(x,y): x^2+y^2\le 1\}$



Interior: $\{x^2+y^2<1\}$ $\qquad \begin{matrix} f'_x(x,y)=2x-1 \\ f'_y(x,y)=4y \end{matrix} \Rightarrow \begin{matrix} x=1/2 \\ y=0 \end{matrix}$ $\quad (1/2,0)$ crit. point.

Boundary: $x^2+y^2=1 \Rightarrow y^2=1-x^2 \Rightarrow g(x) = x^2+2(1-x^2)-x = -x^2-x+2$

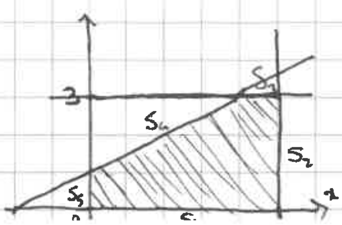
$g'(x)=-2x-1 \Rightarrow -2x-1=0 \Rightarrow x=-\frac{1}{2}$ $\qquad y^2 = 1-\frac{1}{4} = \frac{3}{4} \Rightarrow y=\pm\frac{\sqrt{3}}{2}$

we have two crit. points $\left(-\frac{1}{2},\frac{\sqrt{3}}{2}\right)$ and $\left(-\frac{1}{2},-\frac{\sqrt{3}}{2}\right)$ plus the endpoints $(-1,0)$ and $(1,0)$

Candidates: $\left(\frac{1}{2},0\right), \left(-\frac{1}{2},-\frac{\sqrt{3}}{2}\right), \left(-\frac{1}{2},\frac{\sqrt{3}}{2}\right), (-1,0), (1,0)$

$f\left(\frac{1}{2},0\right) = \frac{1}{4}-\frac{1}{2} = -\frac{1}{4}$ $\quad f\left(-\frac{1}{2},-\frac{\sqrt{3}}{2}\right) = \frac{1}{4}+\cancel{2}\cdot\frac{3}{\cancel{4}2}+\frac{1}{2} = \frac{9}{4}$ $\quad f\left(-\frac{1}{2},\frac{\sqrt{3}}{2}\right) = \frac{9}{4}$ $\quad f(-1,0)=1-1=0$

$f(1,0)=1+1=2$ $\qquad$ MIN at $\left(\frac{1}{2},0\right)$ $\qquad$ MAX at $\left(-\frac{1}{2},\pm\frac{\sqrt{3}}{2}\right)$



3. $f(x,y) = 9x+8y-6(x+y)^2$ $\qquad 0\le x\le 5,\ 0\le y\le 3,\ x\ge 2(y-1)$

We have 5 sides to the boundary. First we consider the interior

Interior: $f'_x(x,y) = 9 - 12(x+y)$ $\quad f'_y(x,y) = 8 - 12(x+y)$ $\quad \begin{cases} 9-12(x+y)=0 \\ 8-12(x+y)=0 \end{cases}$ impossible

There are no crit. points in the interior

Boundary: S_1: $y=0$, $0 \le x \le 5$ $\quad g_1(x) = 9x - 6x^2$ $\quad g'_1(x) = 9 - 12x$

$9 - 12x = 0 \Rightarrow x = \frac{9}{12} = \frac{3}{4}$ we add to candidates $(0,0)$, $(3/4, 0)$ and $(5,0)$

S_2: $x = 5$, $0 \le y \le 3$ $\quad g_2(y) = 45 + 8y - 6(y+5)^2 \Rightarrow g'_2(y) = 8 - 12(y+5) = -12y - 52$

$-12y - 52 = 0 \Rightarrow y = -\frac{52}{12} = -\frac{13}{3}$ not acceptable, we add only $(5,3)$

S_3: to find the range for x, we need the intersection $\begin{cases} x = 2(y-1) \\ y = 3 \end{cases} \Rightarrow \begin{cases} x = 4 \\ y = 3 \end{cases}$

then we have $y = 3$, $4 \le x \le 5$

$g_3(x) = 9x + 24 - 6(3+x)^2 \Rightarrow g'_3(x) = 9 - 12(3+x) = -12x - 27$

$-12x - 27 = 0 \Rightarrow x = -\frac{27}{12} = -\frac{9}{4}$ not acceptable since $4 \le x \le 5 \Rightarrow$ we only add $(4,3)$

S_4: we need also $\begin{cases} x = 2(y-1) \\ x = 0 \end{cases} \Rightarrow \begin{cases} y = 1 \\ x = 0 \end{cases}$ so $x = 2(y-1)$, $1 \le y \le 3$

substitute $x = 2(y-1) \Rightarrow g_4(y) = 9 \cdot 2(y-1) + 8y - 6(2y - 2 + y)^2 = 26y - 18 - 6(3y-2)^2$

$g'_4(y) = 26 - 12(3y-2)3 = 26 - 108y + 72 = 98 - 108y$ $\quad y = \frac{98}{108}$ not acceptable since $y \ge 1$

We add $(0,1)$

S_5: $x = 0$, $0 \le y \le 1$ $\quad g_5(y) = 8y - 6y^2 \Rightarrow g'_5(y) = 8 - 12y \Rightarrow y = \frac{8}{12} = \frac{2}{3}$. We add $(0, 2/3)$

Candidates: $(0,0)$, $(3/4, 0)$, $(5,0)$, $(5,3)$, $(4,3)$, $(0,1)$, $(0, 2/3)$

$f(0,0) = 0$ $\quad f(3/4, 0) = 9 \cdot \frac{3}{4} - \cancel{6}^{3} \cdot \frac{9}{\cancel{16}_{8}} = \frac{27}{8}$ $\quad f(5,0) = 45 - 150 = -105$

$f(5,3) = 45 + 24 - 6 \cdot 64 = -315$ $\quad f(4,3) = 36 + 24 - 6 \cdot 49 = -234$

$f(0,1) = 8 - 6 = 2$ $\quad f(0, 2/3) = \frac{16}{3} - \cancel{6}^{2} \cdot \frac{4}{\cancel{9}_{3}} = \frac{8}{3}$

MIN at $(5,3)$, MAX at $(3/4, 0)$

4. $f(x,y) = ax^2y + bxy + 2xy^2 + c$

(a) f local min at $(\frac{2}{3}, \frac{1}{3})$ with value $-\frac{1}{9}$. $\quad f'_x(x,y) = 2axy + by + 2y^2$

$f'_y(x,y) = ax^2 + bx + 4xy$ $\quad \begin{cases} 2axy + by + 2y^2 = 0 \\ ax^2 + bx + 4xy = 0 \end{cases} \Rightarrow \begin{cases} y(2ax + b + 2y) = 0 \\ x(ax + b + 4y) = 0 \end{cases}$

$\begin{cases} y = 0 \\ x(ax+b) = 0 \end{cases} \Rightarrow \begin{cases} y = 0 \\ x = 0, -b/a \end{cases}$ $\quad (0,0), (-\frac{b}{a}, 0), (0, -\frac{b}{2})$ crit. points

$\begin{cases} 2ax + b + 2y = 0 \\ x(ax + b + 4y) = 0 \end{cases}$

$\begin{cases} b + 2y = 0 \\ x = 0 \end{cases} \Rightarrow \begin{cases} y = -b/2 \\ x = 0 \end{cases}$

$\begin{cases} 2ax + b + 2y = 0 \\ ax + b + 4y = 0 \end{cases} \Rightarrow \begin{cases} 2ax + b + 2y = 0 \\ 2ax + 2b + 8y = 0 \end{cases}$ subtract $\Rightarrow b + 6y = 0 \Rightarrow y = -\frac{b}{6}$

$ax + b + 4 \cdot (-\frac{b}{6}) = 0 \Rightarrow ax + \frac{b}{3} = 0 \Rightarrow x = -\frac{b}{3a}$ $\quad (-\frac{b}{3a}, -\frac{b}{6})$ critical point