Section 12.5

1. (a) $\begin{pmatrix} 0 & -2 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} -1 & 4 \\ 1 & 5 \end{pmatrix} = \begin{pmatrix} 0\cdot(-1)+(-2)\cdot 1 & 0\cdot 4+(-2)\cdot 5 \\ 3\cdot(-1)+1\cdot 1 & 3\cdot 4+1\cdot 5 \end{pmatrix} = \begin{pmatrix} -2 & -10 \\ -2 & 17 \end{pmatrix}$

$\begin{pmatrix} -1 & 4 \\ 1 & 5 \end{pmatrix}\begin{pmatrix} 0 & -2 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} (-1)\cdot 0+4\cdot 3 & (-1)\cdot(-2)+4\cdot 1 \\ 1\cdot 0+5\cdot 3 & 1\cdot(-2)+5\cdot 1 \end{pmatrix} = \begin{pmatrix} 12 & 6 \\ 15 & 3 \end{pmatrix}$

(b) $\begin{pmatrix} 8 & 3 & -2 \\ 1 & 0 & 4 \end{pmatrix}\begin{pmatrix} 2 & -2 \\ 4 & 3 \\ 1 & -5 \end{pmatrix} = \begin{pmatrix} 8\cdot 2+3\cdot 4+(-2)\cdot 1 & 8\cdot(-2)+3\cdot 3+(-2)\cdot(-5) \\ 1\cdot 2+0\cdot 4+4\cdot 1 & 1\cdot(-2)+0\cdot 3+4\cdot(-5) \end{pmatrix} = \begin{pmatrix} 26 & 3 \\ 6 & -22 \end{pmatrix}$

$\begin{pmatrix} 2 & -2 \\ 4 & 3 \\ 1 & -5 \end{pmatrix}\begin{pmatrix} 8 & 3 & -2 \\ 1 & 0 & 4 \end{pmatrix} = \begin{pmatrix} 2\cdot 8+(-2)\cdot 1 & 2\cdot 3+(-2)\cdot 0 & 2\cdot(-2)+(-2)\cdot 4 \\ 4\cdot 8+3\cdot 1 & 4\cdot 3+3\cdot 0 & 4\cdot(-2)+3\cdot 4 \\ 1\cdot 8+(-5)\cdot 1 & 1\cdot 3+(-5)\cdot 0 & 1\cdot(-2)+(-5)\cdot 4 \end{pmatrix} = \begin{pmatrix} 14 & 6 & -12 \\ 35 & 12 & 4 \\ 3 & 3 & -22 \end{pmatrix}$

(c) $\begin{pmatrix} -1 & 0 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ -1 & 1 \\ 0 & 2 \end{pmatrix}$ not possible $\begin{pmatrix} 3 & 1 \\ -1 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 3\cdot(-1)+1\cdot 2 & 3\cdot 0+1\cdot 4 \\ (-1)\cdot(-1)+1\cdot 2 & -1\cdot 0+1\cdot 4 \\ 0\cdot(-1)+2\cdot 2 & 0\cdot 0+2\cdot 4 \end{pmatrix} = \begin{pmatrix} -1 & 4 \\ 1 & 4 \\ 4 & 8 \end{pmatrix}$

(d) $\begin{pmatrix} 0 \\ -2 \\ 4 \end{pmatrix}\begin{pmatrix} 0 & -2 & 3 \end{pmatrix} = \begin{pmatrix} 0\cdot 0 & 0\cdot(-2) & 0\cdot 3 \\ -2\cdot 0 & (-2)\cdot(-2) & -2\cdot 3 \\ 4\cdot 0 & 4\cdot(-2) & 4\cdot 3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 4 & -6 \\ 0 & -8 & 12 \end{pmatrix}$

$\begin{pmatrix} 0 & -2 & 3 \end{pmatrix}\begin{pmatrix} 0 \\ -2 \\ 4 \end{pmatrix} = 0\cdot 0+(-2)\cdot(-2)+3\cdot 4 = 4+12 = 16$

4. (a) $\begin{cases} x_1+x_2=3 \\ 3x_1+5x_2=5 \end{cases} \longrightarrow \begin{pmatrix} 1 & 1 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$

(b) $\begin{cases} x_1+2x_2+x_3=4 \\ x_1-x_2+x_3=5 \\ 2x_1+3x_2-x_3=1 \end{cases} \longrightarrow \begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \\ 2 & 3 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 1 \end{pmatrix}$

(c) $\begin{cases} 2x_1-3x_2+x_3=0 \\ x_1+x_2-x_3=0 \end{cases} \longrightarrow \begin{pmatrix} 2 & -3 & 1 \\ 1 & 1 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Section 12.8

1. (a) $\begin{cases} x_1+x_2=3 \\ 3x_1+5x_2=5 \end{cases}$ pivot $\left(\begin{array}{cc|c} 1 & 1 & 3 \\ 3 & 5 & 5 \end{array}\right) R_2\to R_2-3R_1 \rightarrow \left(\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 2 & -4 \end{array}\right) R_2\to \frac{R_2}{2}$

$\left(\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 1 & -2 \end{array}\right) R_1\to R_1-R_2 \longrightarrow \left(\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & -2 \end{array}\right)$ solution is $x_1=5$, $x_2=-2$

(b) $\begin{cases} x_1+2x_2+x_3=4 \\ x_1-x_2+x_3=5 \\ 2x_1+3x_2-x_3=1 \end{cases}$ pivot $\left(\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 1 & -1 & 1 & 5 \\ 2 & 3 & -1 & 1 \end{array}\right) \begin{matrix} R_2\to R_2-R_1 \\ R_3\to R_3-2R_1 \end{matrix}$ pivot $\left(\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & -3 & 0 & 1 \\ 0 & -1 & -3 & -7 \end{array}\right) R_3\to R_3-\frac{1}{3}R_2$

$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & -3 & 0 & 1 \\ 0 & 0 & -3 & -\frac{22}{3} \end{array}\right) \begin{matrix} R_2\to \frac{R_2}{-3} \\ R_3\to \frac{R_3}{-3} \end{matrix} \longrightarrow \left(\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & 0 & -\frac{1}{3} \\ 0 & 0 & 1 & \frac{22}{9} \end{array}\right) R_1\to R_1-R_3 \longrightarrow \left(\begin{array}{ccc|c} 1 & 2 & 0 & \frac{14}{9} \\ 0 & 1 & 0 & -\frac{1}{3} \\ 0 & 0 & 1 & \frac{22}{9} \end{array}\right) R_1\to R_1-2R_2$

$\left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{20}{9} \\ 0 & 1 & 0 & -\frac{1}{3} \\ 0 & 0 & 1 & \frac{22}{9} \end{array}\right)$ solution: $x_1=\frac{20}{9}$, $x_2=-\frac{1}{3}$, $x_3=\frac{22}{9}$

(c) $\begin{cases} 2x_1 - 3x_2 + x_3 = 0 \\ x_1 + x_2 - x_3 = 0 \end{cases}$ $\left(\begin{array}{ccc|c} 2 & -3 & 1 & 0 \\ 1 & 1 & -1 & 0 \end{array}\right)$ since we have a rectangular system, we turn x_3 into a parameter (we call it t)

$\begin{cases} 2x_1 - 3x_2 = -t \\ x_1 + x_2 = t \end{cases}$ $\left(\begin{array}{cc|c} 2 & -3 & -t \\ 1 & 1 & t \end{array}\right)$ ← pivot $R_2 \to R_2 - \frac{1}{2}R_1$ $\left(\begin{array}{cc|c} 2 & -3 & -t \\ 0 & 5/2 & \frac{3}{2}t \end{array}\right)$ $R_2 \to \frac{2}{5}R_2$

$\left(\begin{array}{cc|c} 2 & -3 & -t \\ 0 & 1 & \frac{3}{5}t \end{array}\right)$ $R_1 \to R_1 + 3R_2$ $\to \left(\begin{array}{cc|c} 2 & 0 & \frac{4}{5}t \\ 0 & 1 & \frac{3}{5}t \end{array}\right)$ $R_1 \to \frac{R_1}{2}$ $\to \left(\begin{array}{cc|c} 1 & 0 & \frac{2}{5}t \\ 0 & 1 & \frac{3}{5}t \end{array}\right)$

solution is $x_1 = \frac{2}{5}t$, $x_2 = \frac{3}{5}t$, $x_3 = t$ for any $t \in \mathbb{R}$.

2. $\begin{cases} x + y - z = 1 \\ x - y + 2z = 2 \\ x + 2y + az = b \end{cases}$ $\left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 1 & -1 & 2 & 2 \\ 1 & 2 & a & b \end{array}\right)$ ← pivot $\begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{array}$ $\left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & -2 & 3 & 1 \\ 0 & 1 & a+1 & b-1 \end{array}\right)$ $R_3 \to R_3 + \frac{1}{2}R_2$

$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & -2 & 3 & 1 \\ 0 & 0 & a+\frac{5}{2} & b-\frac{1}{2} \end{array}\right)$ $R_2 \to \frac{R_2}{-2}$ $\left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & -3/2 & -1/2 \\ 0 & 0 & a+\frac{5}{2} & b-\frac{1}{2} \end{array}\right)$ assume $a \neq -\frac{5}{2}$ then we can divide by $a+\frac{5}{2}$

$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & -3/2 & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{b-\frac{1}{2}}{a+\frac{5}{2}} \end{array}\right)$ $\begin{array}{l} R_1 \to R_1 + R_3 \\ R_2 \to R_2 + \frac{3}{2}R_3 \end{array}$ $\to \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 + \frac{2b-1}{2a+5} \\ 0 & 1 & 0 & -\frac{1}{2} + \frac{2b-1}{2a+5} \\ 0 & 0 & 1 & \frac{2b-1}{2a+5} \end{array}\right)$ $R_1 \to R_1 - R_2$

$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & -\frac{1}{2} + \frac{2b-1}{2a+5} \\ 0 & 0 & 1 & \frac{2b-1}{2a+5} \end{array}\right)$ if $a \neq -\frac{5}{2}$, the solutions are $x_1 = \frac{1}{2}$, $x_2 = -\frac{1}{2} + \frac{2b-1}{2a+5}$, $x_3 = \frac{2b-1}{2a+5}$

if $a = -\frac{5}{2}$ and $b \neq \frac{1}{2}$ there are no solutions.

If $a = -\frac{5}{2}$ and $b = \frac{1}{2}$ we have $\begin{pmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & -\frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix}$ → we turn $x_3 = t$ into a parameter and get

$\left(\begin{array}{cc|c} 1 & 1 & 1+t \\ 0 & 1 & -\frac{1}{2} + \frac{3}{2}t \end{array}\right)$ $R_1 \to R_1 - R_2$ $\left(\begin{array}{cc|c} 1 & 0 & 1+t-\frac{3t-1}{2} \\ 0 & 1 & \frac{3t-1}{2} \end{array}\right) = \left(\begin{array}{cc|c} 1 & 0 & \frac{1-t}{2} \\ 0 & 1 & \frac{3t-1}{2} \end{array}\right)$

3. $\begin{cases} 2w + x + 4y + 3z = 1 \\ w + 3x + 2y - z = 3c \\ w + x + 2y + z = c^2 \end{cases}$ $\left(\begin{array}{cccc|c} 2 & 1 & 4 & 3 & 1 \\ 1 & 3 & 2 & -1 & 3c \\ 1 & 1 & 2 & 1 & c^2 \end{array}\right)$ pivot $\begin{array}{l} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - R_1 \end{array}$ $\left(\begin{array}{cccc|c} 2 & 1 & 4 & 3 & 1 \\ -5 & 0 & -10 & -10 & 3c-3 \\ -1 & 0 & -2 & -2 & c^2-1 \end{array}\right)$ $R_3 \to R_3 - \frac{1}{5}R_2$

$\left(\begin{array}{cccc|c} 2 & 1 & 4 & 3 & 1 \\ -5 & 0 & -11 & -10 & 3c-3 \\ 0 & 0 & 0 & 0 & c^2-1-\frac{3c-3}{5} \end{array}\right)$ $R_3 \to 5R_3$

$\begin{array}{l} R_1 \to R_1 - 4R_3 \\ R_2 \to R_2 + 11R_3 \end{array}$

to have solution we need $c^2 - 1 - \frac{3c-3}{5} = 0$