MATEMATISKA INSTITUTIONEN	Tentamensskrivning i
STOCKHOLMS UNIVERSITET	Matematik III Komplex Analys
Avd. Matematik	7.5 hp
Examinator: Rikard Bögvad	2023/10/23

No calculators, books, or notes allowed.

Each problem is worth 5 points; total 30, grade E attained at 15. There are six problems in total, printed on both sides of the page.

1. Use the residue theorem to evaluate, as a function of $\lambda \in \mathbb{R}$,

$$\int_{-\infty}^{\infty} \frac{2\exp(-2\pi i\lambda x)}{\exp(\pi x)+\exp(-\pi x)}dx$$

2. Evaluate

$$\int_{|z|=1}\int_{|w|=1} \frac{\cos(2\pi wz)}{1-2zw}dwdz$$

3. Determine the Laurent series of

$$f(z) = \frac{2}{z^2-1}$$

in the annulus $r < |z-1| < R$. Your answer should consider finitely many cases depending on the radii R and/or r.

4.

(a) Determine the image of the unit disk $\{z \in \mathbb{C}\ ;\ |z| < 1\}$ under

$$z \mapsto \frac{1+z}{1-z}$$

(b) Consider

$$f(z) = \frac{z}{(1-z)^2}$$

What is the image of the unit disk? Is f a conformal mapping from the disk to its image?

[Hint: compose the mapping from (a) with another transformation.]

5. Consider the function

$$T(z) = \sum_{n=0}^{\infty} 3^{-n^2} z^n$$

(a) Show that, on the circle where $|z| = 9^k$,

$$\left| \sum_{n \neq k} 3^{-n^2} z^n \right| < |3^{-k^2} z^k|$$

(b) How many zeros does T have in the disk $|z| < 9^k$?

6. Suppose f is holomorphic inside and on the circle where $|z| = 1$. Assume $f(0) \neq 0$ and $f(t) \neq 0$ for all t on the circle $|t| = 1$.

(a) Why does f have only a finite number of zeros inside the disk $|z| < 1$?

(b) Show that, if $a_1, \ldots, a_n$ are the zeros of f inside the disk counted with multiplicity, then the function

$$\log \left| \frac{f(z)}{\prod_{k=1}^{n} (z - a_k)} \right|$$

(extended by continuity to $z = a_1, \ldots, a_n$) is a harmonic function.

(c) Deduce that

$$\log |f(0)| = \sum_{k=1}^{n} \log |a_k| + \frac{1}{2\pi} \int_0^{2\pi} \log |f(e^{i\theta})| d\theta$$