MATEMATISKA INSTITUTIONEN	Tentamensskrivning i
STOCKHOLMS UNIVERSITET	Matematik III Komplex Analys
Avd. Matematik	7.5 hp
Examinator: Rikard Bögvad	2023/10/23

No calculators, books, or notes allowed. Each problem is worth 5 points; total 30, grade E attained at 15.

1. Use the residue theorem to evaluate, as a function of $\lambda \in \mathbb{R}$,

$$\int_{-\infty}^{\infty} \frac{2\exp(-2\pi i\lambda x)}{\exp(\pi x)+\exp(-\pi x)}dx$$

Solution. The final answer is

$$\frac{2}{\exp(\pi\lambda)+\exp(-\pi\lambda)}$$

which can be obtained in a few ways: by a rectangular contour that goes around one pole; by a contour that goes around infinitely many poles; or by a semicircular contour after a change of variable. See Figure 1.

Going around one pole. The denominator vanishes when

$$\exp(\pi x)+\exp(-\pi x)=0 \iff \exp(2\pi x)=-1 \iff x\in\frac{i}{2}+i\mathbb{Z}$$

We shift the contour from $\int_{-\infty}^{\infty}$ to take advantage of the pole at $i/2$ (or $-i/2$). Let C be the rectangular curve from $-R$ to R along the real axis, then R to $R+i$ vertically, followed by $R+i$ to $R-i$ horizontally, and finally $-R+i$ to $-R$. The pole $i/2$ lies inside, and there are no others, so

$$\int_C \frac{2\exp(-2\pi i\lambda z)}{\exp(\pi z)+\exp(-\pi z)}dz = 2\pi i\,\mathrm{Res}(\dots, z=i/2)$$

The horizontal parts of C contribute

$$(1+e^{2\pi\lambda})\int_{-R}^{R}\frac{2\exp(-2\pi i\lambda x)}{\exp(\pi x)+\exp(-\pi x)}dx$$

where the factor $e^{2\pi\lambda}$ comes from shifting by i as follows. Since $\exp(\pi i)=\exp(-\pi i)=-1$,

$$\frac{2\exp(-2\pi i\lambda(x+i))}{\exp(\pi(x+i))+\exp(-\pi(x+i))}=\frac{e^{2\pi\lambda}}{-1}\cdot\frac{2\exp(-2\pi i\lambda x)}{\exp(\pi x)+\exp(-\pi x)}$$

This minus sign compensates for the reversed orientation ($-R$ to R on the bottom, but $R+i$ to $-R+i$ on the top, of the rectangle).

The vertical parts of C disappear as $R \to \infty$. Each side has length 1, whereas the integrand tends to 0. The numerator is bounded on the segments $z = \pm R + iy$, $0 \le y \le 1$, since

$$|2\exp(-2\pi i(\pm R + iy))| = 2\exp(2\pi y) \le 2\exp(2\pi)$$

The other terms are exponentially small:

$$\frac{2}{\exp(\pi(\pm R + iy)) + \exp(-\pi(\pm R + iy))} \lesssim \frac{1}{\exp(\pi R)} \to 0.$$

Taking $R \to \infty$, it follows that

$$2\pi i \operatorname{Res}(\dots, z = i/2) = \int_C = (1 + e^{2\pi\lambda}) \int_{-\infty}^{\infty}$$

The answer is of the form

$$\int_{-\infty}^{\infty} \frac{2\exp(-2\pi i\lambda x)}{\exp(\pi x) + \exp(-\pi x)} dx = \frac{2\pi i \operatorname{Res}(\dots, z = i/2)}{1 + e^{2\pi\lambda}}$$

We claim that

$$\operatorname{Res}\left(\frac{2\exp(-2\pi i\lambda z)}{\exp(\pi z) + \exp(-\pi z)}, z = i/2\right) = \frac{1}{\pi i} e^{\pi\lambda}$$

Indeed, recentering at $i/2$, we have

$$\exp(\pi z) + \exp(-\pi z) = \exp(\pi(z - i/2))\exp(\pi i/2) + \exp(-\pi(z - i/2))\exp(-\pi i/2)$$

Since $\exp(\pm\pi i/2) = \pm i$,

$$\exp(\pi z) + \exp(-\pi z) = i\Big(\exp(\pi(z - i/2)) - \exp(-\pi(z - i/2))\Big)$$

Substituting the power series for the exponential function,

$$\exp(\pi z) + \exp(-\pi z) = i2\pi(z - i/2) + \dots$$

It follows that

$$\frac{2}{\exp(\pi z) + \exp(-\pi z)} = \frac{1}{\pi i}(z - i/2)^{-1} + \dots$$

so the residue is as claimed. The numerator $2\exp(-2\pi i\lambda z)$ is holomorphic so we simply evaluate at $z = i/2$ and multiply by $1/(\pi i)$. $\square$

Alternative solution: going around many poles

Suppose we take a rectangle up to $R + iH$ instead of $R + i$, with the same integrand as before:

$$f(z) = \frac{2\exp(-2\pi i\lambda z)}{\exp(\pi z) + \exp(-\pi z)}.$$

The vertical sides can be estimated as before, but now each has length $|H|$ instead of 1. Their contribution is negligible as long as $|H|\exp(-\pi R) \to$

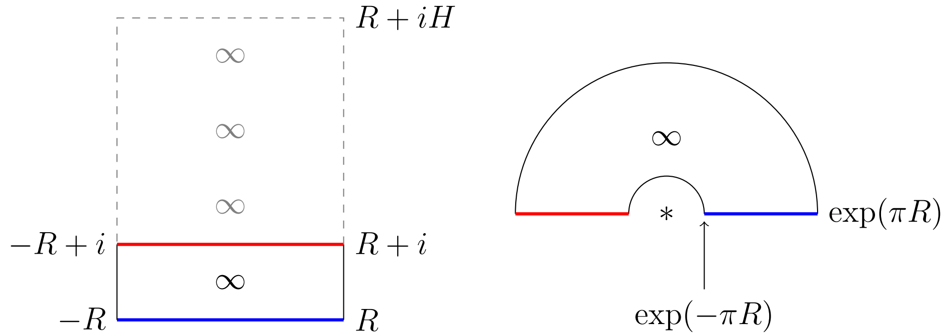

FIGURE 1. Three approaches to Task 1. Left: rectangular contour around $i/2$. Left, gray: rectangle going around multiple poles $z_n = i/2 + in$. Right: an indented semicircle enclosing a pole at i, but avoiding the branch point at 0. These four arcs correspond to the four sides of the original rectangle under $x \mapsto \exp(\pi x)$, which also maps the poles via $i = \exp(\pi i/2)$.

0. We may choose $H = \pm R$ for example. The sign is important because of the remaining side. On the side opposite to the real axis, we have

$$
\begin{aligned}
f(x + iH) &= \frac{2\exp(-2\pi i\lambda(x + iH))}{\exp(\pi(x + iH)) + \exp(-\pi(x + iH))} \\
&= 2e^{2\pi\lambda H} \frac{\exp(-2\pi i\lambda x)}{\exp(\pi x)e^{\pi i H} + \exp(-\pi x)e^{-\pi i H}}
\end{aligned}
$$

which will be exponentially small as $|H| \to \infty$, provided that λ and H have opposite sign. We choose the sign of H to be positive if $\lambda < 0$, or negative if $\lambda > 0$. This method does not help if $\lambda = 0$, but that can be addressed separately. For instance, we may take a limit $\lambda \to 0$ at the end once we have the answer for $\lambda < 0$ (the integrand has exponential decay, so one can justify taking the limit under the integral sign). The integral for $\lambda = 0$ can also be done by finding an antiderivative. Let us therefore restrict to $\lambda \neq 0$. There is no loss of generality in assuming one sign or the other, since the integrand is even under $\lambda \mapsto \pm\lambda$.

One can arrive at the same result using a semicircular contour of radius R instead of a rectangle (for instance, quoting Jordan's lemma to argue that the upper part of the semicircle is negligible). As in the rectangular approach, the semicircle should lie in either the upper or lower half-plane, depending on the sign of λ.

To fix ideas, let us take a rectangle in the upper half-plane. The same logic applies if the sign of λ forces us into the lower half-plane, or if we preferred semicircles to begin with. In any case, we have a growing contour that encloses more and more poles:

$$z_n = \frac{i}{2} + in$$

The residue theorem then implies, in the limit of very large R and H,

$$\int_{-\infty}^{\infty} \frac{2\exp(-2\pi i\lambda x)}{\exp(\pi x) + \exp(-\pi x)} dx = 2\pi i \sum_n \operatorname{Res}\left(f(z), z = \frac{i}{2} + in\right)$$

where the sum is over $n = 0, 1, 2, \ldots$ if $\lambda < 0$ (as we assume), or over $n = -1, -2, -3, \ldots$ if $\lambda > 0$ (if one made the opposite assumption). After computing the residues, we will be left with a geometric series.

We claim

$$\operatorname{Res}(f, z_n) = \frac{1}{\pi i} \exp(\pi\lambda) \cdot (-\exp(2\pi\lambda))^n.$$

Assuming the claim, the sum is

$$\begin{aligned} 2\pi i \sum_{n=0}^{\infty} \operatorname{Res}(f, z_n) &= 2\pi i \cdot \frac{\exp(\pi\lambda)}{\pi i} \sum_{n=0}^{\infty} (-\exp(2\pi\lambda))^n \\ &= 2\exp(\pi\lambda) \frac{1}{1 + \exp(2\pi\lambda)} \end{aligned}$$

by summing a geometric series $\frac{1}{1+q} = \sum_n (-q)^n$. After factoring out $\exp(\pi\lambda)$, we get the final answer $\frac{2}{\exp(\pi\lambda)+\exp(-\pi\lambda)}$ as required.

Finally, let us check that the residues are as claimed. To compute the residue at z_n, we add and subtract z_n to re-center things:

$$f(z) = \frac{2\exp(-2\pi i\lambda z_n)\exp(-2\pi i\lambda(z - z_n))}{\exp(\pi z_n)\exp(\pi(z - z_n)) + \exp(-\pi z_n)\exp(-\pi(z - z_n))}$$

For $z = i/2 + in = i(1/2 + n)$, knowing $\exp(\pi i/2) = i$ and $\exp(\pi i) = -1$, we have

$$\begin{aligned} \exp(\pi z_n) &= \exp(\pi i/2)\exp(\pi i n) = i(-1)^n \\ \exp(-\pi z_n) &= \exp(-\pi i/2)\exp(-\pi i n) = -i(-1)^n \\ \exp(-2\pi i\lambda z_n) &= \exp(2\pi\lambda(1/2 + n)) = \exp(\pi\lambda)\exp(2\pi\lambda n) \end{aligned}$$

Therefore

$$f(z) = \frac{2\exp(\pi\lambda)\exp(2\pi\lambda n)}{i(-1)^n} \cdot \frac{\exp(-2\pi i\lambda(z - z_n))}{\exp(\pi(z - z_n)) - \exp(-\pi(z - z_n))}$$

where we can now substitute the power series $\exp(q) = 1 + q + \ldots$:

$$f(z) = \frac{2\exp(\pi\lambda)\exp(2\pi\lambda n)}{i(-1)^n} \cdot \frac{1 + \ldots}{2\pi(z - z_n) + \ldots}$$

Extracting the coefficient of $(z - z_n)^{-1}$, we see that

$$\operatorname{Res}(f, z_n) = \frac{\exp(\pi\lambda)\exp(2\pi\lambda n)}{i\pi(-1)^n} = \frac{\exp(\pi\lambda)}{\pi i}(-\exp(2\pi\lambda))^n$$

as claimed. $\square$

Alternative solution: change of variable

Another approach that many found tempting was to remove the exponentials by a change of variable:

$$z = \exp(\pi x), \qquad dz = \pi\exp(\pi x)dx = \pi z\ dx \implies$$
$$\int_{-\infty}^{\infty} \frac{2}{e^{\pi x} + e^{-\pi x}} e^{-2\pi i\lambda x} dx = \frac{2}{\pi}\int_0^{\infty} \frac{1}{z^2+1} z^{-2i\lambda} dz$$

(or very similarly with $z = e^{-\pi x}$ instead of $e^{\pi x}$). The good news is that one can more easily find the poles $z = \pm i$. The bad news is that, for non-zero real λ, the power $z^{-2i\lambda}$ is not holomorphic at $z = 0$, but has a branch point.

The integrand is holomorphic inside and on an indented semicircle (as before, either in the upper or lower half-plane). Only one of the poles contributes, for instance $z = i$ if the contour lies in the upper half-plane. The residue can be computed by expanding in powers of $z - i$, say

$$\frac{2}{z^2+1} z^{-2i\lambda} = \frac{2}{(z-i)(2i+z-i)}(i^{-2i\lambda} + \ldots) = \frac{2i^{-2i\lambda}}{2i}(z-i)^{-1} + \ldots$$

Therefore, after we substitute $i = \exp(\pi i/2)$, the residue is

$$\frac{2i^{-2i\lambda}}{2i} = \frac{1}{i}\exp\left(\frac{\pi i}{2}(-2i\lambda)\right) = \frac{1}{i}\exp(\pi\lambda).$$

By the residue theorem, in the limit of a big semicircle with a small indent,

$$\int_0^{\infty} \frac{2}{z^2+1}(z^{-2i\lambda} + (-z)^{-2i\lambda})dz = 2\pi i \operatorname{Res}(\ldots, z = i)$$

We have, from $\exp(\pi i) = -1$,

$$(-z)^{-2i\lambda} = \exp(\pi i(-2i\lambda))z^{-2i\lambda} = \exp(2\pi\lambda)z^{-2i\lambda}$$

Therefore

$$(1+\exp(2\pi\lambda))\int_0^{\infty} \frac{2}{z^2+1} z^{-2i\lambda} dz = 2\pi i \operatorname{Res}(\ldots, z = i) = 2\pi i \cdot \frac{1}{i}\exp(\pi\lambda)$$

Finally, the original integral can be evaluated:

$$\int_{-\infty}^{\infty} \frac{2}{e^{\pi x}+e^{-\pi x}} e^{-2\pi i \lambda x} dx = \frac{1}{\pi}\int_0^{\infty} \frac{2}{z^2+1} z^{-2i\lambda} dz = \frac{1}{\pi} \cdot \frac{\exp(\pi\lambda)}{1+\exp(2\pi\lambda)} \frac{2\pi i}{i}$$

□

Rubric 1. The 5 points correspond more or less closely with success at the following five sub-tasks (roughly 1 point each):

- *Find the poles*
- *Choose a contour*
- *Compute the residue(s)*
- *Bound the integrals along the remaining sides*
- *State the residue theorem carefully*

□

2. Evaluate

$$\int_{|z|=1} \int_{|w|=1} \frac{\cos(2\pi wz)}{1-2zw} dwdz$$

Solution. Let us integrate over w. There is a pole at $w = \frac{1}{2z}$ where $1-2zw = 0$. The pole lies inside the circle because $|\frac{1}{2z}| = \frac{1}{2}$ for any z on the circle $|z| = 1$. Cauchy's integral formula, namely

$$f(w_0) = \frac{1}{2\pi i} \int \frac{f(w)}{w-w_0} dw$$

applied to the function $w \mapsto \cos(2\pi wz)$ and the point $w_0 = 1/(2z)$, gives

$$\int \frac{\cos(2\pi wz)}{1-2zw} dw = \frac{-1}{2z} \int \frac{\cos(2\pi wz)}{w-1/(2z)} dw = -\frac{1}{2z} \cdot 2\pi i \cos(2\pi z \cdot 1/(2z))$$

The same conclusion can also be drawn by calculating the residue at $w = 1/(2z)$. By either method,

$$\int_{|w|=1} \frac{\cos(2\pi wz)}{1-2zw} dw = -\frac{2\pi i}{2z}\cos(\pi) = \frac{\pi i}{z}$$

since $\cos(\pi) = -1$. The integral over z is then

$$\int_{|z|=1} \int_{|w|=1} \frac{\cos(2\pi wz)}{1-2zw} dwdz = \int_{|z|=1} \frac{\pi i}{z} dz = \pi i(2\pi i) = -2\pi^2.$$

□

Rubric 2. There are three main steps:

- *Observation that* $|\frac{1}{2z}| < 1$ *so it contributes to the integral*
- *Cauchy's integral formula for* $\int dw$

- *Cauchy's integral formula for* $\int dz$

Partial credit out of 5 is possible along the lines of $1 + 2 + 2$.

Full credit is awarded for success by other means, e.g. presenting it as a residue calculation. Students could also say something about several variables. □

3. Determine the Laurent series of

$$f(z) = \frac{2}{z^2 - 1}$$

in the annulus $r < |z - 1| < R$. Your answer should consider finitely many cases depending on the radii R and/or r.

Solution. If $R \leq 2$, then the Laurent series is

$$f(z) = \frac{1}{z-1} + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2^{n+1}}(z-1)^n = \sum_{n=-1}^{\infty} \frac{(-1)^{n+1}}{2^{n+1}}(z-1)^n$$

If $R > 2$, then the Laurent series is

$$f(z) = \sum_{n=1}^{\infty}(-1)^n 2^n (z-1)^{-n-1} = \sum_{k=-\infty}^{-2} (-2)^{-k-1}(z-1)^k$$

The denominator is a difference of squares:

$$z^2 - 1 = (z-1)(z+1)$$

There are two singularities 1 and -1. The Laurent series depends on whether the center $z_0 = 1$ is closer to z or to -1. Since $|1 - (-1)| = 2$, we consider the cases $|z - 1| < 2$ and $|z - 1| > 2$ separately.

The function can be written as a partial fraction:

$$f(z) = \frac{2}{z^2 - 1} = \frac{1}{z-1} - \frac{1}{z+1}$$

The first term is already a power $(z-1)^{-1}$ around $z_0 = 1$. The second term can be expanded in a geometric series, using

$$\frac{1}{1+q} = \sum_{n=0}^{\infty}(-1)^n q^n.$$

This converges for $|q| < 1$, which is why the two cases must be treated differently. We choose q accordingly.

If $|z - 1| < 2$, then

$$\frac{1}{z+1} = \frac{1}{z-1+2} = \frac{1}{2} \cdot \frac{1}{1 + (z-1)/2} = \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n 2^{-n}(z-1)^n$$

where the geometric series converges because $|(z-1)/2| < 1$. Subtracting this from the original term gives

$$f(z) = \frac{1}{z-1} - \frac{1}{z+1} = \frac{1}{z+1} - \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n 2^{-n}(z-1)^n$$

This is the expansion as claimed, after a simplification

$$-\frac{1}{2}(-1)^n 2^{-n} = (-1)^{n+1}2^{-n-1}.$$

In the other case, if $|z-1| > 2$, then

$$\frac{1}{z+1} = \frac{1}{z-1}\cdot\frac{1}{1+2/(z-1)} = \frac{1}{z-1}\sum_{n=0}^{\infty}(-1)^n 2^n(z-1)^{-n}$$

This time the series converges because $|2/(z-1)| < 1$. The term $n = 0$ cancels with the other factor from f:

$$f(z) = \frac{1}{z-1} + \frac{1}{z+1} = \frac{1-1}{z-1} + \sum_{n=1}^{\infty}(-2)^n(z-1)^{-n-1}$$

so the series involves negative powers from -2 downward, as claimed. $\square$

Rubric 3. Total 5=1+1+1+2 points roughly for the following

- *Partial fraction*
- *Identifying the poles*
- *Geometric series*
- *Finding the different regions*

Students could also have other approaches worth full credit, e.g. writing an integral formula for the coefficients in the Laurent series, and determining whether the contour goes around other poles.

It's also correct to skip the partial fraction, writing instead

$$\frac{2}{z^2-1} = (z-1)^{-1}\frac{2}{z+1}$$

The factor $1/(z+1)$ *is a geometric series as above, and the other factor* $(z-1)^{-1}$ *just shifts the series by* $(z-1)^k \mapsto (z-1)^{k-1}$. $\square$

4.

(a) Determine the image of the unit disk $\{z \in \mathbb{C}\ ;\ |z| < 1\}$ under

$$z \mapsto \frac{1+z}{1-z}$$

(b) Consider

$$f(z) = \frac{z}{(1-z)^2}$$

What is the image of the unit disk? Is f a conformal mapping from the disk to its image?

[Hint: compose the mapping from (a) with another transformation.]

Solution.

(a) The image is the right half-plane, where $\mathrm{Re}(w) > 0$. We use the fact that Möbius transformations take circles to circles. It is therefore enough to check four points:

$$0 \mapsto 1, \quad 1 \mapsto \infty, \quad -1 \mapsto 0, \quad i \mapsto \frac{1+i}{1-i} = i$$

The points $1, -1, i$ lie on the unit circle, while their images $\infty, 0, i$ lie on the imaginary axis. Therefore the unit disk is either mapped to the left half-plane or the right half-plane, and it must be the right half-plane because $0 \mapsto 1$.

(b) Square the map from (a). We have

$$\left(\frac{1+z}{1-z}\right)^2 - 1 = \frac{(1+z)^2 - (1-z)^2}{(1-z)^2} = \frac{4z}{(1-z)^2}$$

Under squaring, the right-half plane is mapped to $\mathbb{C}\backslash]-\infty, 0]$ because

$$(re^{i\theta})^2 = r^2 e^{2i\theta}$$

In the right-half plane, $-\pi/2 < \theta < \pi/2$, so 2θ attains all values from $-\pi$ to π. Therefore $r^2e^{2i\theta}$ misses only the negative real axis.

If we rescale by a positive factor, $\mathbb{C}\backslash]-\infty, 0]$ is preserved. The sequence of mappings

$$z \mapsto \frac{1+z}{1-z} \mapsto \left(\frac{1+z}{1-z}\right)^2 \mapsto \frac{1}{4}\left(\frac{1+z}{1-z}\right)^2 - \frac{1}{4}$$

takes the unit disk to the half-plane, then $\mathbb{C}\backslash]-\infty, 0]$ then $\mathbb{C}\backslash]-\infty, -1/4]$. The image is the slit region where $]-\infty, -1/4]$ is removed.

Yes, f is a conformal mapping because it is a composition of conformal mappings. Given w, the equation $f(z) = w$ is a quadratic for z, but only one of the roots lies inside the unit disk.

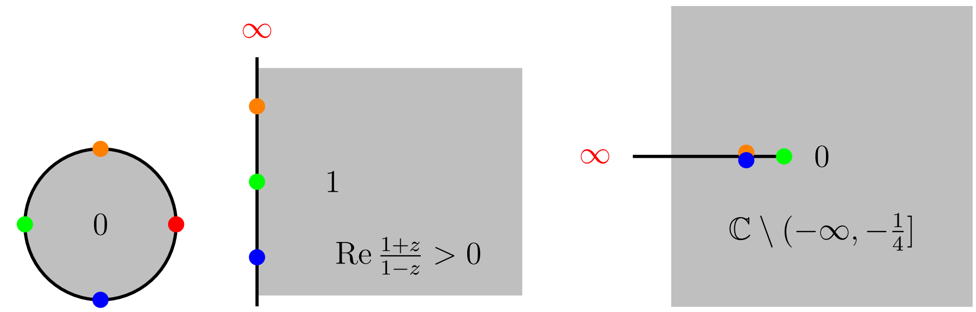

FIGURE 2. (Task 4.) The unit disk is mapped to the right half-plane, then to a slit plane.

It is important that we work with the open unit disk $|z| < 1$. Thus f is a conformal mapping even though it may be two-to-one on the boundary circle. Indeed, i and $-i$ have the same image.

$\square$

Rubric 4. Total 5 points, as follows

- *(a) Möbius transformations take circles to circles*
- *(a) Checking images of 3 points on the unit circle to determine the image of the boundary*
- *(a) Determining image of interior by checking one more point inside (or by considering orientation)*
- *(b) Calculation relating $z/(1-z)^2$ to the mapping from (a), or possibly an alternative approach identifying the image. The image is a slit region $\mathbb{C}\backslash]-\infty, -1/4]$.*
- *(b) Yes, the mapping is one-to-one on the unit disk (even though it is not one-to-one on the closed disk)*

$\square$

5. Consider the function

$$T(z) = \sum_{n=0}^{\infty} 3^{-n^2} z^n$$

(a) Show that, on the circle where $|z| = 9^k$,

$$\left| \sum_{n \neq k} 3^{-n^2} z^n \right| < |3^{-k^2} z^k|$$

(b) How many zeros does T have in the disk where $|z| < 9^k$?

Solution. First let us solve (b) using (a). The function T has precisely k zeros in the disk where $|z| < 9^k$, and we can prove it using Rouché's theorem. Clearly $3^{-k^2} z^k$ is a holomorphic function with k zeros in the disk where $|z| < 9^k$ (namely $z = 0$ is a zero with multiplicity k). The series T and the sum $\sum_{n \neq k}$ are also holomorphic, because the coefficients 3^{-n^2} decay rapidly enough to make the power series converge. Assuming the inequality from part (a), Rouché's theorem implies that T has the same number of zeros as $3^{-k^2} z^k$. The result follows.

To prove (a), we first observe that

$$|3^{-k^2} z^k| = 3^{-k^2} |z|^k = 3^{-k^2} 3^{2k^2} = 3^{k^2}$$

for any z on the circle where $|z| = 9^k = 3^{2k}$. Similarly for $n \neq k$,

$$|3^{-n^2} z^n| = 3^{-n^2+2kn}$$

To compare $n = k$ with the other terms $n \neq k$ in the sum $T(z)$, we use the triangle inequality:

$$\left| \sum_{n \neq k} 3^{-n^2} z^n \right| \leq \sum_{n \neq k} 3^{-n^2+2kn}$$

To compare with the term for k, we add/subtract k^2 and write the exponent as

$$-n^2 + 2kn = k^2 - (k-n)^2$$

Therefore

$$\left| \sum_{n \neq k} 3^{-n^2} z^n \right| \leq \sum_{n \neq k} 3^{-n^2+2kn} = 3^{k^2} \sum_{n \neq k} 3^{-(k-n)^2}$$

and the result follows if we can show that this last sum is less than 1. Indeed it is! Changing variables to $m = k - n$ (for $0 \leq n < k$) or $m = n - k$ (for $n > k$), we have

$$\sum_{n \neq k} 3^{-(k-n)^2} = \sum_{m=1}^{k} 3^{-m^2} + \sum_{m=1}^{\infty} 3^{-m^2} < 2 \sum_{m=1}^{\infty} 3^{-m^2}$$

Since $m^2 \geq m$ (with strict inequality for $m > 1$), we can compare this to a geometric series:

$$\sum_{m=1}^{\infty} 3^{-m^2} < \sum_{m=1}^{\infty} 3^{-m} = \frac{1/3}{1 - 1/3} = \frac{1}{2}$$

Finally,

$$\left|\sum_{n\neq k} 3^{-n^2} z^n\right| \leq \sum_{n\neq k} 3^{-n^2+2kn} = 3^{k^2}\sum_{n\neq k} 3^{-(k-n)^2} < 3^{k^2}\cdot 2\sum_{m=1}^{\infty} 3^{-m^2} < 3^{k^2}$$

which proves (a). □

Rubric 5. Total 5 points, where the points for (b) can be earned without any attempt at (a).

- *(b) statement of Rouché*
- *(b) application of Rouché (what functions, etc)*
- *(b) correct conclusion: k zeros in the disk $|z| < 9^k$*
- *(a) determine $|3^{-n^2} z^n|$ on the circle*
- *(a) set-up to compare $n \neq k$ with k, calculation with geometric series*

□

6. Suppose f is holomorphic inside and on the circle where $|z| = 1$. Assume $f(0) \neq 0$ and $f(t) \neq 0$ for all t on the circle $|t| = 1$.

(a) Why does f have only a finite number of zeros inside the disk $|z| < 1$?

(b) Show that, if $a_1, \dots, a_n$ are the zeros of f inside the disk counted with multiplicity, then the function

$$\log\left|\frac{f(z)}{\prod_{k=1}^{n}(z-a_k)}\right|$$

(extended by continuity to $z = a_1, \dots, a_n$) is a harmonic function.

(c) Deduce that

$$\log|f(0)| = \sum_{k=1}^{n}\log|a_k| + \frac{1}{2\pi}\int_0^{2\pi}\log|f(e^{i\theta})|d\theta$$

Solution.

(a) There are only finitely many zeros in a bounded region because, by the identity theorem, zeros of an analytic function cannot accumulate. We assume $f(0) \neq 0$, so it cannot be that f is identically 0.

(b) Inside the logarithm, $f(z) \div \prod_k (z-a_k)$ is holomorphic and nonzero because $a_1, \dots, a_n$ are all the zeros of f. Therefore there is a holomorphic logarithm, whose real part is the function in question. The real part of a holomorphic function is harmonic.

(c) We apply the mean value property to the harmonic function from part (b). Its value at 0 must be the average over the unit circle:

$$\log\left|\frac{f(0)}{\prod_k(0-a_k)}\right| = \frac{1}{2\pi}\int_0^{2\pi}\log\left|\frac{f(e^{it})}{\prod_k(e^{it}-a_k)}\right|dt$$

Rearranging the logs using $\log(p/q) = \log(p) - \log(q)$ (for real p, q) gives

$$\log|f(0)| - \sum_k \log|a_k| = \frac{1}{2\pi}\int_0^{2\pi}\log|f(e^{it})|dt - \frac{1}{2\pi}\sum_{k=1}^{n}\int_0^{2\pi}\log|e^{it}-a_k|dt$$

This is the desired formula, with one extra sum at the end. We claim that

$$\int_0^{2\pi}\log|e^{it}-a_k|dt = 0$$

for each k.

To prove the claim, since $|e^{it}| = 1$, we can write

$$\log|e^{it}-a_k| = \log|1-a_k e^{-it}|$$

We may take conjugates since $|w| = |\overline{w}|$ for any w, so

$$\log|e^{it}-a_k| = \log|1-\overline{a_k}e^{it}| = \mathrm{Re}(\log(1-\overline{a_k}z)),$$

on the unit circle where $z = e^{it}$. The function $\log|1-\overline{a_k}z|$ is harmonic in the unit disk $|z| < 1$, because it is the real part of a holomorphic function $\log(1-\overline{a_k}z)$ in that domain. We have $1-\overline{a_k}z \neq 0$ in the disk because $|a_k| < 1$, so that $z = 1/\overline{a_k}$ would make $|z| > 1$. Therefore, by the mean value property, the value of this harmonic function at $z = 0$ is equal to its average around the circle:

$$0 = \log(1) = \log|1-\overline{a_k}\cdot 0| = \frac{1}{2\pi}\int_0^{2\pi}\log|1-\overline{a_k}e^{it}|dt$$

This proves the claim. □

Other solutions

Alternatively, one could show the integral is 0 by applying Cauchy's formula (or Cauchy's theorem) for holomorphic functions instead of the mean value property for harmonic functions. Either way, the point is that $|a_k| < 1$ so that one can define a holomorphic logarithm $\log(1-\overline{a_k}z)$. We could also integrate term by term in the power series

$$\log(1-u) = -u - u^2/2 - \dots$$

That gives

$$\int_0^{2\pi} \log(1 - a_k e^{-it})dt = -\sum_{m=1}^{\infty} \frac{1}{m} a_k^m \int_0^{2\pi} e^{-itm} dt = 0$$

where the exponentials for $m \neq 0$ all integrate to 0 by periodicity. Taking the real part shows once again that $\int \log |e^{it} - a_k| dt = 0$ as claimed.

$\square$

Rubric 6. Total 5 points, roughly 1+2+2 for (a)+(b)+(c)

- (a) zeros cannot accumulate
- (b) the real part of an analytic function is harmonic
- (b) dividing f by the product yields a holomorphic function
- (c) mean value property
- (c) $\int_0^{2\pi} \log |e^{it} - a_k| dt = 0$

$\square$