MATEMATISKA INSTITUTIONEN — Tentamensskrivning i
STOCKHOLMS UNIVERSITET — Matematik III Komplex Analys
Avd. Matematik — 7.5 hp
Examinator: Rikard Bögvad — 2023/11/27

- No calculators, books, or notes allowed.
- There are six problems in total, printed on both sides of the page.
- Each problem is worth 5 points; total 30, grade E attained at 15. Show your approach in detail, and state relevant theorems. Partial credit is possible.
- For a question with multiple parts, you can earn credit for part (b) without solving (a). You may use results from the earlier parts to solve the next.

1. [5 points] For $\lambda \in \mathbb{R}$, let

$$f(\lambda) = \int_{-\infty}^{\infty} \exp(-\pi x^2 + 2\pi i \lambda x) dx.$$

Show that

$$f(\lambda) = f(0) \cdot \exp(-\pi \lambda^2).$$

2. [5 points] Evaluate

$$\int_{|z|=1} \int_{|w|=1} \frac{\sin(\pi w z)}{1 - 2zw} dw dz$$

where $|z| = 1$ and $|w| = 1$ refer to the unit circle in each variable, with the usual orientation (anticlockwise).

3. [5 points] Determine all the Laurent series of

$$f(z) = \frac{2}{z^2 + 1}$$

with center $z_0 = i$.

[exam continues on next page]

4. [5 points: 3 points for (a), 2 points for (b)]

(a) Determine the image of the unit disk $\{z \in \mathbb{C}\ ;\ |z| < 1\}$ under

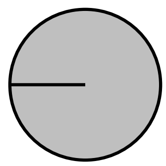
$$z \mapsto \frac{z+i}{iz+1}$$

(b) Let D be the domain

$$\{z \in \mathbb{C}\ ;\ |z| < 1\} \setminus]-1, 0]$$

that is, the unit disk with a segment of the real axis removed. Find a conformal mapping from D onto the upper half-plane $\{w \in \mathbb{C}\ ;\ \mathrm{Im}(w) > 0\}$.

[Hint: compose the mapping from (a) with other transformations such as $w \mapsto w^2$, $\sqrt{w}$, etc.]



5. [5 points]

Determine the number of solutions to $z^5 - z + 1 = 0$ with $|z| < 2$.

6. [5 points: 1 point for (a), 2 points for (b), 2 points for (c)]

Suppose f is holomorphic inside and on the circle where $|z| = 1$. Assume $f(0) \neq 0$ and $f(t) \neq 0$ for all t on the circle $|t| = 1$.

(a) Why does f have only a finite number of zeros inside the disk $|z| < 1$?

(b) Show that, if $a_1, \ldots, a_n$ are the zeros of f inside the disk counted with multiplicity, then the function

$$\log \left| \frac{f(z)}{\prod_{k=1}^{n} (z - a_k)} \right|$$

is a harmonic function in the unit disk.
(This function's values at $z = a_1, \ldots, a_n$ are determined by continuity.)

(c) Deduce that

$$\log |f(0)| = \sum_{k=1}^{n} \log |a_k| + \frac{1}{2\pi} \int_0^{2\pi} \log |f(e^{i\theta})| d\theta.$$