MATEMATISKA INSTITUTIONEN	Tentamensskrivning i
STOCKHOLMS UNIVERSITET	Matematik III Komplex Analys
Avd. Matematik	7.5 hp
Examinator: Rikard Bögvad	2023/11/27

- No calculators, books, or notes allowed.
- There are six problems in total, printed on both sides of the page.
- Each problem is worth 5 points; total 30, grade E attained at 15. Show your approach in detail, and state relevant theorems. Partial credit is possible.
- For a question with multiple parts, you can earn credit for part (b) without solving (a). You may use results from the earlier parts to solve the next.

1. [5 points] For $\lambda \in \mathbb{R}$, let

$$f(\lambda) = \int_{-\infty}^{\infty} \exp(-\pi x^2 + 2\pi i \lambda x)dx.$$

Show that

$$f(\lambda) = f(0) \cdot \exp(-\pi\lambda^2).$$

Solution. We integrate the function

$$g(z) = \exp(-\pi z^2)$$

over a rectangular contour with one side from $-R$ to R, where $R \to \infty$. We choose a rectangle of height λ (or $-\lambda$), either above or below the real axis. We may assume $\lambda \neq 0$, since if $\lambda = 0$, what we are asked to show is simply $f(0) = f(0)$.

Given $\lambda \neq 0$, let C be the contour made of four straight segments oriented from $-R$ to R, then R to $R - i\lambda$ to $-R - i\lambda$ to $-R$. This lies above or below the real axis depending on the sign of λ.

Since g is holomorphic in the whole plane $z \in \mathbb{C}$, it is holomorphic inside and on the contour for all values of R. By Cauchy's theorem,

$$\int_C g(z)dz = 0.$$

We parametrize the base by $-R \leq x \leq R$. The opposite side can be parametrized by $z = x - i\lambda$ with the same range $-R \leq x \leq R$. We

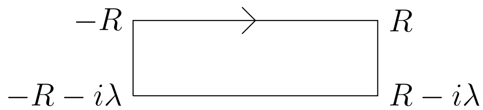


FIGURE 1. Contour for task 1 (drawn in the case $\lambda > 0$).

have

$$g(x - i\lambda) = \exp(-\pi(x - i\lambda)^2) = \exp(-\pi x^2 + 2\pi i x\lambda)\exp(\pi\lambda^2)$$

There is also a sign change from the orientation of the contour. These two horizontal sides contribute

$$\begin{aligned}&\int_{-R}^{R} \exp(-\pi x^2)dx - \int_{-R}^{R} \exp(-\pi x^2 + 2\pi i\lambda x)\exp(\pi\lambda^2)dx \\ &\to f(0) - f(\lambda)\exp(\pi\lambda^2) \qquad (R \to \infty)\end{aligned}$$

The vertical sides are negligible as $R \to \infty$. We can parametrize them by $z = R + iy$ and $z = -R + iy$. The range is either $0 \leq y \leq |\lambda|$ or $|\lambda| \leq y \leq 0$, depending on the sign of λ. In any case, these segments each have length $|\lambda|$. To bound g, observe that

$$g(\pm R + iy) = \exp(-(\pm R + iy)^2) = \exp(-R^2 + y^2)\exp(\mp 2\pi i R y)$$

so

$$|g(\pm R + iy)| \leq \exp(-R^2 + \lambda^2).$$

It follows that

$$\left| \int_{\text{vertical sides}} g \right| \leq 2|\lambda|\exp(-R^2 + \lambda^2) \to 0$$

as $R \to \infty$, for any given value of λ fixed in advance. The factor 2 arises because there are two vertical sides estimated the same way.

Going back to Cauchy's theorem, as $R \to \infty$, the integral becomes

$$0 = \int_{\text{horizontal}} + \int_{\text{vertical}} \to f(0) - f(\lambda)\exp(\pi\lambda^2) + 0.$$

Since $\exp(\pi\lambda^2) \neq 0$, we may divide and obtain

$$f(\lambda) = f(0) \cdot \exp(-\pi\lambda^2).$$

□

Remark: if it helps, there is no loss of generality in assuming $\lambda > 0$. One can change variables $x \mapsto -x$ in the integral to show that

$$f(-\lambda) = f(\lambda)$$

so it is enough to consider only one of the possible signs of λ.

□

2. [5 points] Evaluate

$$\int_{|z|=1}\int_{|w|=1}\frac{\sin(\pi wz)}{1-2zw}dwdz$$

where $|z|=1$ and $|w|=1$ refer to the unit circle in each variable, with the usual orientation (anticlockwise).

Solution. The final answer is

$$\int_{|z|=1}\int_{|w|=1}\frac{\sin(\pi wz)}{1-2zw}dwdz = 2\pi^2.$$

By Cauchy's integral formula, applied to $\sin(\pi wz)$ as a function of w, we have

$$\int\frac{\sin(\pi wz)}{1-2wz}dw = \frac{-1}{2z}\int\frac{\sin(\pi wz)}{w-1/(2z)}dw = \frac{-1}{2z}\cdot 2\pi i\sin(\pi z\cdot\frac{1}{2z})$$

Alternatively, the same conclusion can be drawn from the residue theorem. Either way

$$\int\frac{\sin(\pi wz)}{1-2wz}dw = \frac{-1}{2z}2\pi i\sin(\pi/2) = \frac{-\pi i}{z}.$$

The integral over z is then

$$\int\frac{-\pi i}{z}dz = -\pi i\cdot 2\pi i = 2\pi^2.$$

The final step $\int z^{-1}dz = 2\pi i$ can be justified by Cauchy's integral formula again, or by the residue theorem, or by direct evaluation. □

3. [5 points] Determine all the Laurent series of

$$f(z) = \frac{2}{z^2+1}$$

with center $z_0 = i$.

Solution. There are two Laurent series, one of them convergent for $|z-i|<2$ and the other convergent for $|z-i|>2$. For $|z-i|<2$, we have

$$f(z) = \sum_{m=-1}^{\infty}\frac{1}{2}\left(\frac{i}{2}\right)^m(z-i)^m$$

and for $|z-i|>2$,

$$f(z) = \sum_{m=-\infty}^{-2}-\frac{1}{2}\left(\frac{i}{2}\right)^m(z-i)^m.$$

These series can be obtained either by partial fractions, or by factoring. In either approach, one writes

$$\frac{1}{z+i} = \frac{1}{2i+z-i}$$

and expands in a geometric series

$$\frac{1}{1+q} = 1 - q + q^2 + \ldots = \sum_{n=0}^{\infty} (-1)^n q^n.$$

The geometric series converges for $|q| < 1$. If $|z-i| < 2$, then we take $q = \frac{z-i}{2i}$ and expand as follows:

$$\frac{1}{2i+z-i} = \frac{1}{2i} \cdot \frac{1}{1+\frac{z-i}{2i}} = \frac{1}{2i} \sum_{n=0}^{\infty} (-1)^n \left(\frac{z-i}{2i}\right)^n.$$

If $|z-i| > 2$, then we take $q = \frac{2i}{z-i}$ instead:

$$\frac{1}{2i+z-i} = \frac{1}{z-i} \cdot \frac{1}{1+\frac{2i}{z-i}} = \frac{1}{z-i} \sum_{n=0}^{\infty} (-1)^n \left(\frac{2i}{z-i}\right)^n$$

So far, we have a series expansion of $\frac{1}{z+i}$. Some algebra is needed to turn it into a series for f. A direct approach is to write

$$f(z) = \frac{2}{z^2+1} = \frac{2}{(z-i)(z+i)}$$

Then we can simply multiply the series for $\frac{1}{z+i}$ by $\frac{2}{z-i}$. If $|z-i| < 2$, then the result is

$$\begin{aligned} f(z) &= \frac{2}{z-i} \cdot \frac{1}{2i} \cdot \frac{1}{1+\frac{z-i}{2i}} \\ &= \frac{1}{i}(z-i)^{-1} \sum_{n=0}^{\infty} \frac{1}{2^n} \left(\frac{-1}{i}\right)^n (z-i)^n \end{aligned}$$

We can simplify this using

$$\frac{-1}{i} = i$$

and shifting the sum to start from $(z-i)^{-1}$. That is

$$\begin{aligned} f(z) &= \frac{1}{i}(z-i)^{-1} \sum_{n=0}^{\infty} \frac{1}{2^n} \left(\frac{-1}{i}\right)^n (z-i)^n \\ &= \sum_{n=0}^{\infty} 2^{-n} i^{n-1} (z-i)^{n-1} \end{aligned}$$

and we index the sum by $m = n - 1$ instead of n, with $2^{-n} = 2^{-m-1}$. That gives the series as claimed for this case:

$$f(z) = \sum_{m=-1}^{\infty} 2^{-m-1} i^m (z-i)^m = \sum_{m=-1}^{\infty} \frac{1}{2}\Big(\frac{i}{2}\Big)^m (z-i)^m.$$

In the other case, we instead have

$$\begin{aligned} f(z) &= \frac{2}{z-i} \cdot \frac{1}{z-i} \cdot \frac{1}{1+\frac{2i}{z-i}} \\ &= 2(z-i)^{-2} \sum_{n=0}^{\infty} (-1)^n \Big(\frac{2i}{z-i}\Big)^n \end{aligned}$$

The series goes from $(z-i)^{-2}$ downward, with infinitely many negative powers, so we change from n to $m = -n-2$. That gives

$$f(z) = 2 \sum_{m=-\infty}^{-2} (z-i)^m (-2i)^{-m-2}$$

We can simplify by writing

$$f(z) = 2(-2i)^{-2} \sum_{m=-\infty}^{-2} (z-i)^m (-2i)^{-m}$$

and then

$$2(-2i)^{-2} = -2^{-1}, \qquad (-2i)^{-m} = \Big(\frac{-1}{2i}\Big)^m = \Big(\frac{i}{2}\Big)^m.$$

As claimed,

$$f(z) = -\frac{1}{2} \sum_{m=-\infty}^{-2} \Big(\frac{i}{2}\Big)^m (z-i)^m.$$

Partial fraction. Another approach is to write f as a partial fraction:

$$f(z) = \frac{1}{i}\Big(\frac{1}{z-i} - \frac{1}{z+i}\Big)$$

We can then substitute the series for $\frac{1}{z+i}$. The other term $\frac{1}{z-i}$ only adjusts the terms of order $(z-i)^{-1}$ in the series.

If $|z-i| < 2$, then

$$\begin{aligned} f(z) &= \frac{1}{i}(z-i)^{-1} - \frac{1}{i} \cdot \frac{1}{2i} \cdot \frac{1}{1+\frac{z-i}{2i}} \\ &= \frac{1}{i}(z-i)^{-1} + \frac{1}{2} \sum_{m=0}^{\infty} (-1)^m \Big(\frac{z-i}{2i}\Big)^m \end{aligned}$$

This agrees with the other method. The summands for $m \geq 0$ are the same, and the summand $m = -1$ above appears in this approach from the other term $\frac{1}{z-i}$ in the partial fraction.

The other case $|z - i| > 2$ involves some cancellation to make the series start from $(z - i)^{-2}$, despite the term $(z - i)^{-1}$ in the partial fraction. We have

$$\begin{aligned} f(z) &= \frac{1}{i}(z-i)^{-1} - \frac{1}{i} \cdot \frac{1}{z-i} \cdot \frac{1}{1 + \frac{2i}{z-i}} \\ &= \frac{1}{i}(z-i)^{-1} - \frac{1}{i}(z-i)^{-1} \sum_{n=0}^{\infty} (-1)^n \left(\frac{2i}{z-i}\right)^n \end{aligned}$$

where the terms in $(z - i)^{-1}$ cancel. Using again $-\frac{1}{i} = i$ leaves

$$f(z) = i(z-i)^{-1} \sum_{n=1}^{\infty} (-2i)^n (z-i)^{-n}.$$

We can let $m = -n - 1$ and write the final answer as

$$f(z) = \sum_{m=-\infty}^{-2} -\frac{1}{2}\left(\frac{i}{2}\right)^m (z-i)^m.$$

$\square$

4. [5 points: 3 points for (a), 2 points for (b)]
   (a) Determine the image of the unit disk $\{z \in \mathbb{C}\ ;\ |z| < 1\}$ under

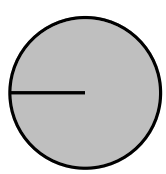
$$z \mapsto \frac{z+i}{iz+1}$$

   (b) Let D be the domain

$$\{z \in \mathbb{C}\ ;\ |z| < 1\} \setminus]-1, 0]$$

   that is, the unit disk with a segment of the real axis removed. Find a conformal mapping from D onto the upper half-plane $\{w \in \mathbb{C}\ ;\ \mathrm{Im}(w) > 0\}$.

   [Hint: compose the mapping from (a) with other transformations such as $w \mapsto w^2$, $\sqrt{w}$, etc.]



Solution.

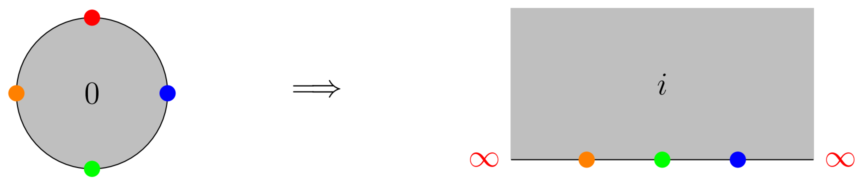


Figure 2. Task 4(a): the transformation takes the unit disk to the upper half-plane.

(a) Möbius transformations take circles to circles. First we compute the images of three points on the unit circle – the unique circle through those points must be the boundary of the image domain. The image might lie either "inside" or "outside" this boundary, so we check one more point from the interior. Alternatively, one can figure it out from the fact that Möbius transformations preserve orientation.

Let us take 1, i, and -1 as boundary points ($-i$ is more than enough, but we'll check that one too as an alternative).

$$\begin{aligned} 1 &\mapsto \frac{1+i}{i+1} = 1 \\ i &\mapsto \frac{2i}{0} = \infty \\ -1 &\mapsto \frac{-1+i}{-i+1} = -1 \\ -i &\mapsto 0. \end{aligned}$$

From any three of these, we see that the image circle is the real axis. Going around the unit circle from -1 to $-i$ to 1, the unit disk is to the left. The image domain must therefore also lie to the left when going from -1 to 0 to 1. It follows that the image is the upper half-plane $\{w \in \mathbb{C} \ ; \ \mathrm{Im}(w) > 0\}$ rather than the lower half-plane.

It would also suffice to observe that

$$0 \mapsto \frac{0+i}{i \cdot 0 + 1} = i,$$

which lies in the upper half-plane.

(b) We take a square root (the principal one) to map the domain to a semicircle. Using the Möbius transformation from (a), we can map the semicircle to a quadrant. Finally, squaring maps the quadrant to the upper half-plane as required.

It is useful to note that the transformation from (a) can be written as

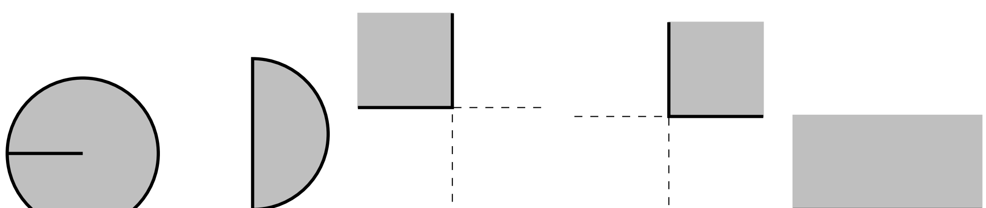
$$w = \frac{z+i}{iz+1} = \frac{1}{i} \cdot \frac{z+i}{z-i}$$

Checking 0, i, $-i$ and 1 from the boundary of the semicircle, we find that $z \mapsto \frac{z+i}{z-i}$ takes them to

$$0 \mapsto -1, \qquad i \mapsto \infty, \qquad 1 \mapsto i, \qquad -i \mapsto 0.$$

This gives the second quadrant (the points $x + iy$ where $x < 0$, $y > 0$). The factor $1/i$ from $w = 1/i \cdot (z+i)/(z-i)$ rotates this back to the first quadrant. Squaring takes the first quadrant to the upper half-plane.

□



z $\quad z_1 = \sqrt{z}$ $\quad z_2 = \frac{z_2+i}{z_2-i}$ $\quad z_3 = z_2/i$ $\quad z_4 = z_3^2$

5. [5 points]

Determine the number of solutions to $z^5 - z + 1 = 0$ with $|z| < 2$.

Solution. There are 5 solutions in the given disk.

Let

$$f(z) = z^5 - z + 1$$

We use Rouché's theorem to compare f to

$$g(z) = z^5.$$

Their difference satisfies

$$|f(z) - g(z)| = |-z + 1| \leq 1 + |z|.$$

We claim

$$|f(z) - g(z)| < |g(z)|$$

We estimate $f - g$ as above, and write $|g(z)| = |z|^5$. It is enough to show

$$1 + |z| < |z|^5$$

which indeed holds true on the circle $|z| = 2$ because $1 + 2 < 2^5$. □

6. [5 points: 1 point for (a), 2 points for (b), 2 points for (c)] Suppose f is holomorphic inside and on the circle where $|z| = 1$. Assume $f(0) \neq 0$ and $f(t) \neq 0$ for all t on the circle $|t| = 1$.

(a) Why does f have only a finite number of zeros inside the disk $|z| < 1$?

(b) Show that, if $a_1, \ldots, a_n$ are the zeros of f inside the disk counted with multiplicity, then the function

$$\log \left| \frac{f(z)}{\prod_{k=1}^{n}(z - a_k)} \right|$$

is a harmonic function in the unit disk.
(This function's values at $z = a_1, \ldots, a_n$ are determined by continuity.)

(c) Deduce that

$$\log |f(0)| = \sum_{k=1}^{n} \log |a_k| + \frac{1}{2\pi} \int_0^{2\pi} \log |f(e^{i\theta})| d\theta.$$

Solution.

(a) There are only finitely many zeros in a bounded region because, by the identity theorem, zeros of an analytic function cannot accumulate. We assume $f(0) \neq 0$, so it cannot be that f is identically 0.

(b) Inside the logarithm, $f(z) \div \prod_k (z - a_k)$ is holomorphic and nonzero because $a_1, \ldots, a_n$ are all the zeros of f. Therefore there is a holomorphic logarithm, whose real part is the function in question. The real part of a holomorphic function is harmonic.

(c) We apply the mean value property to the harmonic function from part (b). Its value at 0 must be the average over the unit circle:

$$\log \left| \frac{f(0)}{\prod_k (0 - a_k)} \right| = \frac{1}{2\pi} \int_0^{2\pi} \log \left| \frac{f(e^{it})}{\prod_k (e^{it} - a_k)} \right| dt$$

Rearranging the logs using $\log(p/q) = \log(p) - \log(q)$ (for real p, q) gives

$$\log |f(0)| - \sum_k \log |a_k| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(e^{it})| dt - \frac{1}{2\pi} \sum_{k=1}^{n} \int_0^{2\pi} \log |e^{it} - a_k| dt$$

This is the desired formula, with one extra sum at the end. We claim that

$$\int_0^{2\pi} \log |e^{it} - a_k| dt = 0$$

for each k.

To prove the claim, since $|e^{it}| = 1$, we can write

$$\log|e^{it} - a_k| = \log|1 - a_k e^{-it}|$$

We may take conjugates since $|w| = |\overline{w}|$ for any w, so

$$\log|e^{it} - a_k| = \log|1 - \overline{a_k} e^{it}| = \mathrm{Re}(\log(1 - \overline{a_k} z)),$$

on the unit circle where $z = e^{it}$. The function $\log|1 - \overline{a_k} z|$ is harmonic in the unit disk $|z| < 1$, because it is the real part of a holomorphic function $\log(1 - \overline{a_k} z)$ in that domain. We have $1 - \overline{a_k} z \neq 0$ in the disk because $|a_k| < 1$, so that $z = 1/\overline{a_k}$ would make $|z| > 1$. Therefore, by the mean value property, the value of this harmonic function at $z = 0$ is equal to its average around the circle:

$$0 = \log(1) = \log|1 - \overline{a_k} \cdot 0| = \frac{1}{2\pi} \int_0^{2\pi} \log|1 - \overline{a_k} e^{it}| dt$$

This proves the claim. $\square$

Other solutions

Alternatively, one could show the integral is 0 by applying Cauchy's formula (or Cauchy's theorem) for holomorphic functions instead of the mean value property for harmonic functions. Either way, the point is that $|a_k| < 1$ so that one can define a holomorphic logarithm $\log(1 - \overline{a_k} z)$. We could also integrate term by term in the power series

$$\log(1 - u) = -u - u^2/2 - \dots$$

That gives

$$\int_0^{2\pi} \log(1 - a_k e^{-it}) dt = -\sum_{m=1}^{\infty} \frac{1}{m} a_k^m \int_0^{2\pi} e^{-itm} dt = 0$$

where the exponentials for $m \neq 0$ all integrate to 0 by periodicity. Taking the real part shows once again that $\int \log|e^{it} - a_k| dt = 0$ as claimed.

$\square$