MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Avd. Matematik
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Tentamensskrivning i
Matematik III Komplex Analys
7.5 hp
2024/10/26

No calculators, books, or notes allowed.

Each problem is worth 5 points; total 30, grade E attained at 15. There are six problems in total, printed on both sides of the page.

1. Given $t \in \mathbb{R}$, use the residue theorem to evaluate

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} \exp(-2\pi i t x) dx.$$

Solution. The final answer is

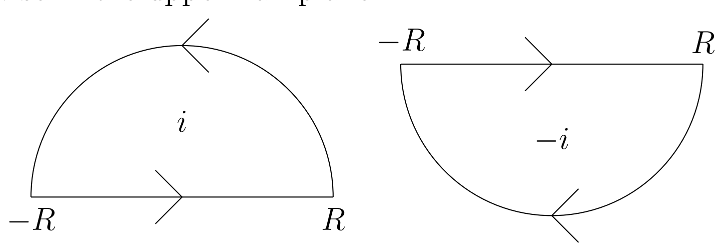
$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} \exp(-2\pi i t x) dx = \pi \exp(-2\pi |t|).$$

One can either treat separate cases for $t \geq 0$ and $t < 0$, or observe from the beginning that the final answer must be an even function with respect to $\pm t$. In the latter approach, it is enough to treat only one of the two cases in detail.

The answer can be deduced from the residue theorem. We consider the function

$$f(z) = \frac{1}{1+z^2} \exp(-2\pi i t z).$$

We integrate f over a semicircular contour. Let C be the segment from $-R$ to R, together with a semicircle of radius R, where $R > 1$ is a large parameter. The semicircle should lie either in the lower half-plane if t is positive or in the upper half-plane if t is negative (both choices work if $t = 0$). The circular part is oriented clockwise in the lower half-plane, or clockwise in the upper half-plane.



The function f is analytic except for poles at $z = \pm i$. Only one of these lies inside the semicircle. To find the residue, write

$$f(z) = \frac{1}{(z-i)(z+i)} \exp(-2\pi i t z).$$

The residue is

$$\operatorname{Res}(f(z), z = \pm i) = \frac{1}{\pm 2i} \exp(\pm 2\pi t).$$

We have chosen the contour so that $\pm$ is the sign opposite to t. That means $\pm t = -|t|$ in all three cases ($t > 0$, $t < 0$, or $t = 0$). Therefore the residue is

$$\operatorname{Res}(f(z), z = \pm i) = \pm\frac{1}{2i} \exp(-2\pi|t|)$$

Recall that the usual statement of the residue theorem assumes the path of integration is oriented anticlockwise. If C goes clockwise instead, this introduces another sign. By the residue theorem,

$$\pm \int_C f(z)dz = 2\pi i \operatorname{Res}(f(z), z = \pm i)$$

The sign cancels so that, in both cases,

$$\int_C f(z)dz = \pi \exp(-2\pi|t|).$$

We claim the integral over the circular part of C vanishes as $R \to \infty$. On the circle where $|z| = R$,

$$|1 + z^2| \geq |z|^2 - 1 = R^2 - 1$$

by the triangle inequality. If $z = x + iy$, then

$$\exp(-2\pi i t z) = \exp(-2\pi i t x) \exp(2\pi t y).$$

Assuming t and y have opposite signs, it follows that

$$|\exp(-2\pi i t z)| \leq 1.$$

Therefore, on the semicircle where $|z| = R$ and $\operatorname{Im}(z)$ has the sign opposite to t, we have a bound

$$|f(z)| \leq \frac{1}{R^2 - 1}.$$

The circular part of C has length πR. By the triangle inequality for integrals,

$$\left| \int f(z)dz \right| \leq \pi R \cdot \frac{1}{R^2 - 1} \to 0$$

as $R \to \infty$. This shows that, as $R \to \infty$, only the integral over the linear part of C survives:

$$\pi \exp(-2\pi|t|) = \int_C f(z)dz \to \int_{-\infty}^{\infty} f(x)dx.$$

$\square$

Rubric for Task 1. The 5 points for this task are awarded for

- Choosing a relevant function $f(z)$
- Choosing a contour (depending on the sign of t)
- Finding the poles and residues
- Relating the desired real integral to a complex contour integral. In particular, some estimates are needed to show that the circular part of the contour can be ignored as $R \to \infty$. This can be done by hand, or quoting results such as "Jordan's lemma".
- Stating the residue theorem or other results used

$\square$

2. Evaluate
$$\int_{|z|=1} \int_{|w|=1} \frac{\exp(2\pi wz)}{1-4zw} dwdz$$
where both circles are oriented anticlockwise.

Solution. Apply Cauchy's integral formula to the integral over w.
$$\frac{\exp(2\pi wz)}{1-4zw} = -\frac{1}{4z} \frac{\exp(2\pi wz)}{w - 1/(4z)}$$
For any z on the circle $|z| = 1$, we have $|1/(4z)| = 1/4$ so this point lies inside the contour of integration with respect to w. The function $w \mapsto \exp(2\pi wz)$ is holomorphic, so by Cauchy's formula
$$\int \frac{\exp(2\pi wz)}{w - 1/(4z)} dw = 2\pi i \exp(2\pi \frac{1}{4z} \cdot z)$$

Next the integral over z can be evaluated. Either by direct parametrization, or by Cauchy's formula again, or by the residue theorem, one has
$$\int \frac{1}{z} dz = 2\pi i.$$
In total,
$$\begin{aligned}\int_{|z|=1} \int_{|w|=1} \frac{\exp(2\pi wz)}{1-4zw} dwdz &= \int -\frac{1}{4z} 2\pi i \exp(\pi/2) dz \\ &= -\frac{1}{4}(2\pi i)^2 \exp(\pi/2) \\ &= \pi^2 \exp(\pi/2).\end{aligned}$$

$\square$

Rubric for Task 2. This task offers a choice between using the residue theorem or Cauchy's formula. Either way, partial credit toward the 5 points may be awarded for

- Stating Cauchy's formula, the residue theorem, or any results you use
- Commenting on why $w = \frac{1}{4z}$ lies inside the contour
- Calculating the residue if you used the residue theorem, or rewriting the function in a way that allows Cauchy's formula to be applied
- Evaluating the outer integral $\int z^{-1}dz$ by any preferred means (for example: parametrizing $z = e^{i\theta}$, or quoting it as a known result, or using the residue theorem, or Cauchy's formula)
- Saying something about analytic/holomorphic functions of two complex variables (Cauchy's formula for a polydisk)

□

3. Calculate all the Laurent series of
$$f(z) = \frac{2}{z(z^2-1)}$$
around the point $z_0 = 2$.

[Hint: your answer should take a different form in each of a finite number of annular domains.]

Solution. There are four different Laurent series, corresponding to the annular domains
$$0 < |z - z_0| < 1, \quad 1 < |z - z_0| < 2, \quad 2 < |z - z_0| < 3, \quad 3 < |z - z_0|.$$
This is because $f(z)$ has singularities $z = 1, 0, -1$ lying on their respective circles $|z - z_0| = 1$, $|z - z_0| = 2$, $|z - z_0| = 3$. Because f has no singularity at z_0, one can also write the innermost domain as a disk $|z - z_0| < 1$ rather than a punctured disk $0 < |z - z_0| < 1$.

The partial fraction for f is
$$f(z) = \frac{2}{z(z^2-1)} = -\frac{2}{z} + \frac{1}{z-1} + \frac{1}{z+1}$$
With respect to the center $z_0 = 2$, this is
$$f(z) = -\frac{2}{(z-2)+2} + \frac{1}{(z-2)+1} + \frac{1}{(z-2)+3}$$
The three parts can all be written as geometric series, but we need to know the size of $z - 2$ compared to the other terms $+2$, $+1$, or $+3$ in the denominator. Recall that the geometric series is
$$\frac{1}{1-q} = 1 + q + q^2 + \ldots = \sum_{n=0}^{\infty} q^n$$

which converges for $|q| < 1$. It is also convenient to note the alternating version, which follows after replacing q by $-q$:

$$\frac{1}{1+q} = 1 - q + q^2 + \ldots = \sum_{n=0}^{\infty} (-1)^n q^n$$

again for $|q| < 1$. To expand a term $\frac{1}{(z-z_0)+a}$, we apply these formulas with either $q = (z - z_0)/a$ or $q = a/(z - z_0)$ depending on which choice makes $|q| < 1$.

The next step is to write each contribution to f in the form $\frac{1}{1+q}$ where $|q| < 1$. If $0 < |z - 2| < 1$, then

$$f(z) = -\frac{1}{1 + (z-2)/2} + \frac{1}{1 + (z-2)} + \frac{1/3}{1 + (z-2)/3}$$

If $1 < |z - 2| < 2$, then

$$f(z) = -\frac{1}{1 + (z-2)/2} + \frac{1/(z-2)}{1 + 1/(z-2)} + \frac{1/3}{1 + (z-2)/3}$$

If $2 < |z - 2| < 3$, then

$$f(z) = -\frac{2/(z-2)}{1 + 2/(z-2)} + \frac{1/(z-2)}{1 + 1/(z-2)} + \frac{1/3}{1 + (z-2)/3}$$

If $3 < |z - 2|$, then

$$f(z) = -\frac{2/(z-2)}{1 + 2/(z-2)} + \frac{1/(z-2)}{1 + 1/(z-2)} + \frac{1/(z-2)}{1 + 3/(z-2)}$$

Finally, the Laurent expansion in each annulus is one combination or another of the following geometric series:

$$\begin{aligned}
\frac{1}{1+(z-2)/2} &= \sum_{n=0}^{\infty}(-1)^n 2^{-n}(z-2)^n \\
\frac{1}{1+(z-2)} &= \sum_{n=0}^{\infty}(-1)^n (z-2)^n \\
\frac{1/3}{1+(z-2)/3} &= \frac{1}{3}\sum_{n=0}^{\infty}(-1)^n 3^{-n}(z-2)^n \\
\frac{1/(z-2)}{1+1/(z-2)} &= (z-2)^{-1}\sum_{n=0}^{\infty}(-1)^n (z-2)^{-n} \\
\frac{2/(z-2)}{1+2/(z-2)} &= 2(z-2)^{-1}\sum_{n=0}^{\infty}(-1)^n 2^n (z-2)^{-n} \\
\frac{1/(z-2)}{1+3/(z-2)} &= (z-2)^{-1}\sum_{n=0}^{\infty}(-1)^n 3^n (z-2)^{-n}
\end{aligned}$$

If $0 < |z-2| < 1$, then

$$\begin{aligned}
f(z) &= -\frac{1}{1+(z-2)/2} + \frac{1}{1+(z-2)} + \frac{1/3}{1+(z-2)/3} \\
&= -\sum_{n=0}^{\infty}(-1)^n 2^{-n}(z-2)^n + \sum_{n=0}^{\infty}(-1)^n (z-2)^n + \frac{1}{3}\sum_{n=0}^{\infty}(-1)^n 3^{-n}(z-2)^n
\end{aligned}$$

Only positive exponents $n \geq 0$ occur, and the total is

$$f(z) = \sum_{n=0}^{\infty}(-1)^n(1-2^{-n}+3^{-n-1})(z-2)^n.$$

If $1 < |z-2| < 2$, then we have to rewrite one of the three terms:

$$\begin{aligned}
f(z) &= -\frac{1}{1+(z-2)/2} + \frac{1/(z-2)}{1+1/(z-2)} + \frac{1/3}{1+(z-2)/3} \\
&= -\sum_{n=0}^{\infty}(-1)^n 2^{-n}(z-2)^n + \sum_{n=0}^{\infty}(-1)^n (z-2)^{-n-1} + \frac{1}{3}\sum_{n=0}^{\infty}(-1)^n 3^{-n}(z-2)^n
\end{aligned}$$

In this case, both positive and negative exponents occur. We have

$$f(z) = \sum_{n\in\mathbb{Z}} a_n (z-2)^n, \quad a_n = \begin{cases} (-1)^{n+1}2^{-n} + (-1)^n 3^{-n-1} & \text{if } n \geq 0 \\ (-1)^{n+1} & \text{if } n < 0 \end{cases}$$

If $2 < |z-2| < 3$, then

$$f(z) = -\frac{2/(z-2)}{1+2/(z-2)} + \frac{1/(z-2)}{1+1/(z-2)} + \frac{1/3}{1+(z-2)/3}$$

$$= -2\sum_{n=0}^{\infty}(-1)^n 2^n (z-2)^{-n-1} + \sum_{n=0}^{\infty}(-1)^n (z-2)^{-n-1} + \frac{1}{3}\sum_{n=0}^{\infty}(-1)^n 3^{-n}(z-2)^n$$

Again, both positive and negative exponents occur. This time

$$f(z) = \sum_{n\in\mathbb{Z}} a_n (z-2)^n, \quad a_n = \begin{cases} (-1)^n 3^{-n-1} & \text{if } n \geq 0 \\ (-1)^n (2^{-n} - 1) & \text{if } n < 0 \end{cases}$$

If $3 < |z-2|$, then

$$f(z) = -\frac{2/(z-2)}{1+2/(z-2)} + \frac{1/(z-2)}{1+1/(z-2)} + \frac{1/(z-2)}{1+3/(z-2)}$$

$$= -2\sum_{n=0}^{\infty}(-1)^n 2^n (z-2)^{-n-1} + \sum_{n=0}^{\infty}(-1)^n (z-2)^{-n-1} + \sum_{n=0}^{\infty}(-1)^n 3^n (z-2)^{-n-1}$$

In this outermost case, there are only negative exponents:

$$f(z) = \sum_{n=0}^{\infty}(-1)^n(-2^{n+1} + 1 + 3^n)(z-2)^{-n-1}$$

□

Rubric for Task 3. The 5 points for this task are awarded for

- Annuli (finding the domains where the Laurent series takes different forms)
- Partial fraction
- Geometric series
- Adding the series together, simplifying the answer
- Recalling relevant theorems or definitions (for example, the integral formula for the coefficients of a Laurent series)

□

4.

(a) Under the transformation

$$z \mapsto \frac{z-1}{z+1}$$

determine the image of the half-plane $\{z \in \mathbb{C}\ ;\ \mathrm{Re}(z) > 0\}$.

(b) How many solutions are there to

$$(z-1)^3 + 3(z-1)^2(z+1) + (z+1)^3 = 0$$

subject to the constraint $\mathrm{Re}(z) > 0$?

Solution.

(a) The image is the unit disk. We can check where the transformation takes 0, i, and ∞ (three points on the boundary).

$$\begin{aligned} 0 &\mapsto -1 \\ i &\mapsto i \\ \infty &\mapsto 1 \end{aligned}$$

The unit circle is the unique circle/line through the images -1, i, and 1. The image of the half-plane must therefore be either the inside ($|w| < 1$) or the outside ($|w| > 1$). Checking one more point in the interior, we find

$$1 \mapsto 0$$

with $|0| < 1$. It follows that, as claimed, the image is the unit disk.

Alternatively, consider the orientation of a path going upward from 0 to i to ∞, with the region $\{\mathrm{Re}(z) > 0\}$ lying to the right. The image of the path goes clockwise from -1 to i to 1. To keep on the right of this path, the image of $\{\mathrm{Re}(z) > 0\}$ must therefore be the unit disk.

(b) The solutions are given by

$$w^3 + 3w^2 + 1 = 0$$

where

$$w = \frac{z-1}{z+1}.$$

We are free to divide by $z+1$ because $z = -1$ is not one of the solutions:

$$(z-1)^3 + 3(z-1)^2(z+1) + (z+1)^3 = -8 \neq 0$$

(and even if $z = -1$ did solve the equation, it would not satisfy the constraint $\mathrm{Re}\, z > 0$).

By part (a), the constraint $\mathrm{Re}(z) > 0$ corresponds to $|w| < 1$. Using Rouché's Theorem, we can show that this polynomial has 2 roots in the unit circle, the same number as $3w^2$. Let $g(w) = w^3 + 3w^2 + 1$ be the given polynomial, and let $f(w) = 3w^2$. On the curve $|w| = 1$, we have

$$|f(w) - g(w)| = |w^3 + 1| \leq 2 < 3 = |f(w)|.$$

Both polynomials are holomorphic inside, with no roots on the circle (clear for $3w^2$, and also true for $w^3 + 3w^2 + 1$ by the triangle inequality). Therefore f and g have the same number of zeros in the unit disk, that is, 2 zeros since $w = 0$ is a double root of $3w^2$.

□

Rubric for Task 4. The 5 points are split into 2 points for (a) and 3 points for (b). Partial credit can be earned from various steps:

- Möbius transformations take circles to circles
- Finding that the image is the unit disk.
- Using the transformation from (a) to set up (b)
- Statement of Rouché's theorem
- Correct application of Rouché's theorem to show the equation has 2 solutions subject to the constraint.

□

5. Define two operators acting on smooth functions by

$$L_1(h) = \frac{\partial}{\partial x}h + i\frac{\partial}{\partial y}h, \quad L_2(h) = \frac{\partial}{\partial x}h - i\frac{\partial}{\partial y}h$$

where $h : D \to \mathbb{C}$ is a function with continuous derivatives of all orders on some domain D.

(a) Show that $L_1 \circ L_2 = L_2 \circ L_1$ where $\circ$ denotes composition.
(b) Show that h is harmonic if and only if $L_1(L_2(h)) = 0$.
(c) Let $h(x, y) = (x - iy)^{2024} + \exp(x)\cos(y) + i\exp(x)\sin(y)$. Is h harmonic?

Solution.

(a) Suppose h is a smooth function on D. In particular, suppose h has continuous derivatives up to second order. That guarantees the equality of mixed partials

$$h_{xy} = h_{yx}$$

where we write subscripts to denote the partial derivative with respect to the indicated variable. By definition of L_1 and L_2,

$$\begin{aligned} L_1\big((L_2(h)\big) &= L_1(h_x - ih_y) \\ &= (h_x - ih_y)_x + i(h_x - ih_y)_y \\ &= h_{xx} - ih_{yx} + ih_{xy} + h_{yy} \end{aligned}$$

using $i^2 = -1$ at the last step. The mixed partials cancel out, leaving

$$L_1\big((L_2(h)\big) = h_{xx} + h_{yy}$$

Composing in the opposite order,

$$\begin{aligned} L_2\big((L_1(h)\big) &= L_2(h_x + ih_y) \\ &= (h_x + ih_y)_x - i(h_x + ih_y)_y \\ &= h_{xx} + ih_{yx} - ih_{xy} + h_{yy} \end{aligned}$$

where the same cancellation happens as before. Therefore

$$L_2\big((L_1(h)\big) = h_{xx} + h_{yy} = L_1\big((L_2(h)\big).$$

This shows that $L_1 \circ L_2 = L_2 \circ L_1$ as operators on smooth functions. □

(b) By definition, h is harmonic if and only if it solves Laplace's equation $h_{xx} + h_{yy} = 0$. From the proof of part (a), we know that $L_1\big(L_2(h)\big) = 0$ is an equivalent way to write Laplace's equation (at least for smooth functions, but smoothness of h is given from the start of the problem). □

(c) Yes, h is harmonic. It can be checked directly that $h_{xx}+h_{yy} = 0$. There is also a shortcut involving L_1 and L_2. In terms of a complex variable $z = x + iy$, we have

$$\begin{aligned} h(x, y) &= (x - iy)^{2024} + \exp(x)\cos(y) + i\exp(x)\sin(y) \\ &= \overline{z}^{2024} + \exp(z) \end{aligned}$$

It follows from the Cauchy–Riemann equations for $\exp(z)$ that

$$L_1 \exp(z) = 0.$$

Taking the conjugate of the Cauchy–Riemann equations for z^{2024} gives

$$L_2\overline{z}^{2024} = 0.$$

We can combine these using the fact that L_1 and L_2 are linear operators. We also know $L_1 \circ L_2 = L_2 \circ L_1$ from part (a). That gives

$$\begin{aligned} L_1\big(L_2(h)\big) &= L_1\big(L_2(\overline{z}^{2024})\big) + L_2\Big(L_1\big(\exp(z)\big)\Big) \\ &= L_1(0) + L_2(0) \\ &= 0. \end{aligned}$$

This shows that h is harmonic, using part (b).

Rubric for Task 5. The 5 points are achieved as 2+1+2 for (a)+(b)+(c). If $L_1\big(L_2(h)\big)$ is miscalculated, partial credit is possible from

- Defining harmonic functions in (b)
- Computing derivatives of the function h from (c)

□

6. Let f be a function holomorphic on $\mathbb{C}$ except for poles, and satisfying

$$f(z+1) = f(z+i) = f(z)$$

for all $z \in \mathbb{C}$.

Let S be the square with corners 0, 1, i, and $1+i$. Let C be the boundary of S oriented anticlockwise. Assume f is not identically 0, and that f does not have any zeros or poles on C.

For p a zero or pole of f, define $m_p(f)$ as follows. If p is a zero of f, let $m_p(f)$ be its order. If p is a pole of f, let $-m_p(f)$ be the order of the pole.

(a) Show that

$$\sum_p m_p(f) = 0$$

where the sum is taken over all zeros and poles of f inside S.

(b) What are the poles and residues of $zf'(z)/f(z)$?

(c) Show that the real and imaginary parts of

$$\sum_p m_p(f) \cdot p$$

are both integers $(0, \pm 1, \pm 2, \ldots)$.

Solution.

(a) By the argument principle,

$$\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} dz = \sum_p m_p(f).$$

It is enough to show the integral is 0.

We claim that the integrand f'/f has the same periodicity as f. Differentiating both sides of $f(z+1) = f(z)$ shows that $f'(z+1) = f'(z)$, and similarly for the other translation $z \mapsto z+i$. This shows that f' is also periodic with respect to the same translations as f. Therefore f'/f is also periodic.

The square C has two pairs of opposite sides, where f'/f takes the same values but the orientation is reversed. Each pair therefore cancels, leaving

$$\int_C \frac{f'(z)}{f(z)} dz = 0.$$

(b) The poles of $zf'(z)/f(z)$ are the poles of f together with the zeros of f. There is some question of whether a pole of f'/f at $z = 0$ could cancel with the other factor z, leaving a removable singularity. However, we are assuming f has no poles on C, so in particular $z = 0$ is not a pole.

The residue is $m_p(f) \cdot p$. We already know from the proof of the argument principle that f'/f has residue given by the multiplicity, and the extra factor p is the value of z at the point p.

(c) By the residue theorem, taking part (b) as a hint,

$$\frac{1}{2\pi i} \int_C z \frac{f'(z)}{f(z)} dz = \sum_p m_p(f) \cdot p.$$

The sides of C can be parametrized by $z = x$ ($0 \le x \le 1$), then $z = 1 + iy$ ($0 \le y \le 1$), then $z = x + i$ with x going backwards, and finally a downward trajectory $z = iy$. We must not forget factors of i from $dz = idy$ on the vertical sides. Integrating over the square one side at a time gives

$$\begin{aligned}\int_C z \frac{f'(z)}{f(z)} dz = \int_0^1 x \frac{f'(x)}{f(x)} dx + i \int_0^1 (1+iy) \frac{f'(1+iy)}{f(1+iy)} dy \\ - \int_0^1 (x+i) \frac{f'(x+i)}{f(x+i)} dx - i \int_0^1 iy \frac{f'(iy)}{f(iy)} dy\end{aligned}$$

As in the solution for (a), we know that f'/f is periodic under $x \mapsto x + i$ and $iy \mapsto iy + 1$. Therefore

$$\int_C z \frac{f'(z)}{f(z)} dz = \int_0^1 i \frac{f'(x)}{f(x)} dx + i \int_0^1 \frac{f'(iy)}{f(iy)} dy.$$

These integrals can be evaluated in terms of logarithms, using the antiderivatives

$$\frac{f'(x)}{f(x)} = \frac{d}{dx} \log f(x), \quad i \frac{f'(iy)}{f(iy)} = \frac{d}{dy} \log f(iy).$$

We have assumed that f has no zeros on C, so it is possible to define logarithms of $f(x)$ and $f(iy)$. Integration then gives

$$\int_C z \frac{f'(z)}{f(z)} dz = i \log f(x) \Big|_{x=0}^{x=1} + \log f(iy) \Big|_{y=0}^{y=1}$$

There are many choices of logarithm, which differ from each other by an additive constant that disappears under d/dx or d/dy, but may return after integration. By periodicity, $f(x)$ takes the same value at $x = 1$ and $x = 0$. Therefore

$$\log f(x = 1) - \log f(x = 0) = 2\pi i a$$

for some integer a. Likewise $f(iy)$ takes the same value at both endpoints, so the logarithms can only differ by $2\pi ib$ for some integer b.

It follows that, for some integers a and b,

$$\int_C z\frac{f'(z)}{f(z)}dz = i \cdot 2\pi ia + 2\pi ib = 2\pi i(b + ia).$$

Combining this with the residue calculation from above, we find

$$\sum_p m_p(f) \cdot p = b + ia$$

where both the real and imaginary parts are integers.

Rubric for Task 6. The 5 points correspond to

- Stating the argument principle
- Observing that opposite sides of the square cancel because f is doubly periodic
- The poles of $zf'(z)/f(z)$ are the poles as well as the zeros of f
- The residues are $m_p(f) \cdot p$
- Using properties of logarithms and periodicity of f to complete the proof of (c)

□

All the rubrics:
Rubric for Task 1: The 5 points for this task are awarded for

- Choosing a relevant function $f(z)$
- Choosing a contour (depending on the sign of t)
- Finding the poles and residues
- Relating the desired real integral to the complex contour integral. In particular, some estimates are needed to show that the circular part of the contour can be ignored as $R \to \infty$. This can be done by hand, or quoting results such as "Jordan's lemma".
- Stating the residue theorem or other results used

Rubric for Task 2: This task offers a choice between using the residue theorem or Cauchy's formula. Either way, partial credit toward the 5 points may be awarded for

- Stating Cauchy's formula, the residue theorem, or any results you use
- Commenting on why $w = \frac{1}{4z}$ lies inside the contour
- Calculating the residue if you used the residue theorem, or rewriting the function in a way that allows Cauchy's formula to be applied

- Evaluating the outer integral $\int z^{-1}dz$ by any preferred means (for example: parametrizing $z = e^{i\theta}$, or quoting it as a known result, or using the residue theorem, or Cauchy's formula)
- Saying something about analytic/holomorphic functions of two complex variables (Cauchy's formula for a polydisk)

Rubric for Task 3: The 5 points for this task are awarded for

- Annuli (finding the domains where the Laurent series takes different forms)
- Partial fraction
- Geometric series
- Adding the series together, simplifying the answer
- Recalling relevant theorems or definitions (for example, the integral formula for the coefficients of a Laurent series)

Rubric for Task 4: The 5 points are split into 2 points for (a) and 3 points for (b). Partial credit can be earned from various steps:

- Möbius transformations take circles to circles
- Finding that the image is the unit disk.
- Using the transformation from (a) to set up (b)
- Statement of Rouché's theorem
- Correct application of Rouché's theorem to show the equation has 2 solutions subject to the constraint.

Rubric for Task 5: The 5 points are achieved as 2+1+2 for (a)+(b)+(c). If $L_1\big(L_2(h)\big)$ is miscalculated, partial credit is possible from

- Defining harmonic functions in (b)
- Computing derivatives of the function h from (c)

Rubric for Task 6: The 5 points correspond to

- Stating the argument principle
- Observing that opposite sides of the square cancel because f is doubly periodic
- The poles of $zf'(z)/f(z)$ are the poles as well as the zeros of f
- The residues are $m_p(f) \cdot p$
- Using properties of logarithms and periodicity of f to complete the proof of (c)