

Instructions:

- During the exam you may not use any book, notes, calculators or any other supporting material.
- There are six problems in total, printed on both sides of the page.
- In all solutions, justify your answers. An adequate amount of details are required for full marks.
- Use natural language, not just mathematical symbols. Write clearly and legibly.
- Begin each problem (1–6) on a new page and place the problem number at the top of the page. Only write on the front side of the page. For problems with multiple parts, you do not need to start each part on a separate page.

Grades: Each solved problem is awarded up to 10 points. At least 30 points guarantee grade E, 36 for D, 42 for C, 48 for B, and 54 for A. Note that the problems are not ordered according to the difficulty!

1. (a) Let $\Omega \subset \mathbb{C}$ be an open set, and let $a \in \Omega$. Define what it means for a holomorphic function on $\Omega \setminus \{a\}$ to have at $z = a$:
 - i. a removable singularity; ii. a pole of order k ; iii. an essential singularity.
- (b) Prove that a function $f(x + iy) = u(x, y) + iv(x, y)$, with u and v real-valued C^2 functions in a connected open set $\Omega \subset \mathbb{C}$, is holomorphic in Ω if and only if it satisfies the Cauchy–Riemann equations.

Solution (a) See Definition 8, pp. 278-279 in the coursebook.

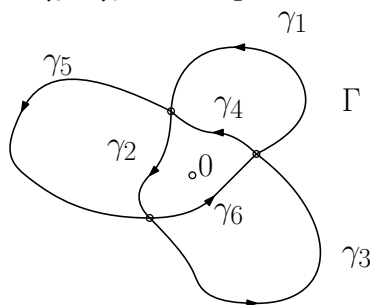
(b) See Theorem 4 and Theorem 5, in pp. 73-75 of the coursebook.

2. (a) Consider the contour $\Gamma = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 + \gamma_5 + \gamma_6$ of the figure below.

Calculate

$$\int_{\Gamma} \frac{\cos z}{z} dz.$$

Motivate your answer fully.



- (b) Given $t > 0$, calculate, using a suitable contour, the integral

$$\int_{-\infty}^{+\infty} \frac{e^{itx}}{(1+x^2)^2} dx.$$

Solution (a) (The following is only one possible argument) Note that we can write

$$\Gamma = (\gamma_1 + \gamma_5 + \gamma_3) + (\gamma_4 + \gamma_2 + \gamma_6) =: \Gamma_1 + \Gamma_2,$$

with Γ_1, Γ_2 simple closed contours with 0 on their interior. Since $\cos z$ is entire, by the Cauchy theorem

$$\int_{\Gamma} \frac{\cos z}{z} dz = \sum_{j=1}^2 \int_{\Gamma_j} \frac{\cos z}{z} dz = 4\pi i.$$

(b) Letting

$$f(z) = \frac{e^{itz}}{(1+z^2)^2},$$

we note that f is a meromorphic function that has only poles of order 2 at both $z_0 = \pm i$. For $R > 2$, let Γ_R be the semicircular contour, positively oriented, consisting on the segment $[-R, R]$ and the upper half circle $C_R := \{z : z = Re^{it}, t \in [0, \pi]\}$. The contour Γ_R contains the pole $z_0 = i$ on its interior. By the Residue theorem we have that

$$2\pi i \text{Res}(f, z_0) = \int_{\Gamma_R} f(z) dz = \int_{-R}^R f(z) dz + \int_{C_R} f(z) dz.$$

To calculate the residue, notice that

$$\begin{aligned} \text{Res}(f, z_0) &= \lim_{z \rightarrow z_0} \frac{d}{dz} ((z-i)^2 f(z)) = \lim_{z \rightarrow z_0} \frac{d}{dz} \left(\frac{e^{itz}}{(z+i)^2} \right) \\ &= \frac{it e^{itz_0}}{(z_0+i)^2} - 2 \frac{e^{itz_0}}{(z_0+i)^3} = \left(-\frac{it}{4} - \frac{1}{-4i} \right) e^{-t} = -i \frac{1+t}{4} e^{-t}. \end{aligned}$$

Thus

$$\int_{\Gamma_R} f(z) dz = \pi \frac{t+1}{2} e^{-t}.$$

Notice that for $z \in C_R$, we have that

$$|f(z)| \leq \frac{e^{-t \text{Im} z}}{(R^2 - 1)^2} \leq \frac{1}{(R^2 - 1)^2}.$$

This yields

$$\left| \int_{C_R} f(z) dz \right| \leq \frac{\pi R}{(R^2 - 1)^2},$$

which implies that

$$\lim_{R \rightarrow +\infty} \int_{C_R} f(z) dz = 0.$$

In turn, this gives

$$\int_{-\infty}^{+\infty} \frac{e^{itx}}{(1+x^2)^2} dx = \pi \frac{1+t}{2} e^{-t}.$$

3. (a) Prove that any Mobius transformation maps circles and lines in either a circle or a line.
- (b) Determine the image of the upper half plane

$$\mathbb{H} := \{z \in \mathbb{C} : \text{Im}(z) > 0\}$$

under the mapping

$$z \mapsto \frac{z+i}{iz+1}.$$

Motivate your answer.

Solution (a) See Section 7.4 in the course literature.

(b) We notice that T is a Mobius transformation and that, if we call T this mapping, it satisfies

$$T(0) = i, \quad T(1) = 1, \quad T(\infty) = -i.$$

So T maps bijectively the real line to the unit circle. Moreover, since orientation is preserved, it maps the upper half space, that lays to the left of real line with the orientation given by $0, 1, \infty$ to the left of the unit circle with orientation given by $i, 1, -i$. So, the image of the upper half plane is the unbounded component of the complementary of the unit circle.

4. Show that $z^4 + 12z + 1$ has exactly three complex zeros with $1 < |z| < 4$.

Solution The idea is to use Rouché's theorem twice. One to see that all 4 roots are inside the disc of centre zero and radius 4, and another one to show that in the unit disc there is only one.

Let $p(z) = z^4 + 12z + 1$. Let $f(z) = z^4$. We know that f has 4 zeroes inside $|z| < 4$. Also on $|z| = 4$

$$|f(z) - p(z)| \leq 12 \cdot 4 + 1 < 4^4 = |f(z)|.$$

So Rouché's theorem yields that p has also 4 zeroes inside the same disc.

Let now $g(z) = 12z + 1$. This function has exactly one zero inside the unit disc. Moreover on $|z| = 1$ we have that, by the reverse triangle inequality

$$|g(z) - p(z)| = 1 < 12 - 1 = 12|z| - 1 \leq |12z + 1| = |g(z)|.$$

So p has one zero inside of the unit disc.

Notice that if for some $|z_0| = 1$, $p(z_0) = 0$, the previous inequality would give us that

$$1 = |g(z_0)| < |g(z_0)|,$$

which would lead to a contradiction.

Please, turn page over $\longrightarrow$

5. (a) Let $\Omega \subset \mathbb{C}$ be an open, non-empty and connected set, and let $\lambda \in \mathbb{C}$ fixed. Prove that a function $f : \Omega \rightarrow \mathbb{C}$ satisfies

$$f'(z) = \lambda f(z), \quad \forall z \in \Omega,$$

if, and only if, there exists a constant $c \in \mathbb{C}$ such that $f(z) = ce^{\lambda z}$.

Hint: You may consider useful studying the function $g(z) = e^{-\lambda z} f(z)$.

- (b) Determine whether there exists an entire functions h with $h(0) = 0$ satisfying that

$$|h(z)| < 2|h'(z)|, \quad \forall z \in \mathbb{C}.$$

If such function exists, find them all.

Solution (a) Letting $g(z) = e^{-\lambda z} f(z)$, we have that g is holomorphic in Ω . Moreover

$$g'(z) = -\lambda g(z) + e^{-\lambda z} f'(z) = 0.$$

Since Ω is open and connected, this yields that g must be constant on Ω , from where the result follows.

- (b) Let assume that h is such a function. By the inequality assumed, it follows that $h'(z) \neq 0$ for all $z \in \mathbb{C}$. Thus, the function

$$g(z) := \frac{h(z)}{h'(z)},$$

must be an entire function. The inequality yields also that g is bounded, and so, by Liouville's theorem, it must be constant. Since $h(0) = 0$, that constant must be zero, and so, h is constant equal zero. But this would contradict the inequality assumed.

So we reach the conclusion that such function can not exists.

6. Let $\{a_n\}_{n \geq 1}$ be the sequence given by

$$a_n = \begin{cases} 1 & \text{if } n = 2^k \text{ for some } k \in \mathbb{N} \\ n^{-n} & \text{otherwise} \end{cases}$$

Consider the series

$$f(z) = \sum_{n=1}^{\infty} a_n z^n.$$

(a) Prove that

$$\inf_{n \geq 1} \left(\sup_{m \geq n} |a_m|^{1/m} \right) = 1.$$

Determine the largest domain on which the above power series defines a holomorphic function.

(b) Compute the contour integral

$$\int_{|z - \frac{1}{4}| = \frac{1}{2}} \frac{e^{f(z)}}{z^2} dz,$$

where the contour is oriented positively.

Solution (a) Since given any integer n , there exists an integer $m = 2^k \geq n$ for some $k \geq 1$ (it suffices taking $k = \lceil \log_2 n \rceil$), a direct calculation shows the first statement.

The expression gives that $\limsup_n |a_n|^{1/n} = 1$, so by the definition of the radius of convergence we know that the series defines a holomorphic function on the unit disk, and that the series diverges for $|z| > 1$.

(b) Since f is holomorphic in $\mathbb{D}$, so it is $e^{f(z)}$. Notice that the given contour is the circle centred at $\frac{1}{4}$ of radius $\frac{1}{2}$, which is fully contained in the unit disk. We can then use the Cauchy-representation formula to say that the value of the integral equals

$$2\pi i \frac{d}{dz} (e^{f(z)})|_{z=0} = 2\pi i \left(e^{f(0)} f'(0) \right) = 2\pi i.$$

where we have used that $f(0) = 0$, $f'(0) = a_1 = 1$.