1. The conditions give the following set of equations

$$\begin{cases} f(1) = b+1 \\ f'(1) = b \end{cases}$$

which gives

$$\begin{cases} a = b+1 \\ 3a+1 = b. \end{cases}$$

This is a system of two equations that is to be solved for a and b. By substituting the first into the second equation we obtain

$$3(b+1)+1 = b$$

$$3b+4 = b$$

$$2b = -4$$

Hence $b=-2$. Since $a=b+1$ this also gives $a=-1$. So the solution is $a=-1$ and $b=-2$.

2. (a) The integral is computed by integration by parts applied twice.

$$\int_0^1 (x-1)^2 \mathrm{e}^x \mathrm{d}x = (x-1)^2\mathrm{e}^x\Big|_{x=0}^1 - 2\int_0^1 (x-1)\mathrm{e}^x\mathrm{d}x$$
$$= -1 - 2\,(x-1)\mathrm{e}^x\Big|_{x=0}^1 + 2\int_0^1 \mathrm{e}^x\mathrm{d}x$$
$$-1-2+2\mathrm{e}-2 = 2\mathrm{e}-5.$$

(b) First compute the definite integral by substituting $u=\ln x$. Then

$$\int \frac{1}{x}(\ln x)^a \mathrm{d}x = \int u^a \mathrm{d}u = \begin{cases} \frac{1}{a+1}u^{a+1}+C & \text{if } a \neq -1, \\ \ln a + C, & \text{if } a=-1 \end{cases}$$

with $C \in \mathbb{R}$. Then

$$\int_{\mathrm{e}}^{\infty} \frac{1}{x}(\ln x)^a \mathrm{d}x = \int_1^{\infty} u^a \mathrm{d}u = \begin{cases} \frac{1}{a+1}, & \text{if } a < -1, \\ +\infty, & \text{if } a \geq -1. \end{cases}$$

Hence the integral converges for $a < -1$.

3. The matrix is invertible if and only if the determinant does not vanish. Then the determinant is computed

$$\det\begin{pmatrix} a & 1 & 2 \\ 0 & 2 & 1 \\ 1 & a & 1 \end{pmatrix} = a\det\begin{pmatrix} 2 & 1 \\ a & 1 \end{pmatrix} - 0*\det\begin{pmatrix} 1 & 2 \\ a & 1 \end{pmatrix} + \det\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = a(2-a)-0+3 = -a^2+2a-3$$

Since there are no real zeros of the quadratic polynomial, the matrix is always invertible for every $a \in \mathbb{R}$.

4. The formula for the Taylor polynomial is

$$f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 + \frac{1}{6}f'''(0)x^3$$

Then we compute

$$f'(x) = \frac{1}{2}\frac{1+2x}{(1+x+x^2)^{1/2}}\mathrm{e}^{(1+x+x^2)^{1/2}}$$

$$f''(x) = -\frac{1}{4}\frac{(1+2x)^2}{(1+x+x^2)^{3/2}}\mathrm{e}^{(1+x+x^2)^{1/2}} + \frac{1}{(1+x+x^2)^{1/2}}\mathrm{e}^{(1+x+x^2)^{1/2}} + \frac{1}{4}\frac{(1+2x)}{1+x+x^2}\mathrm{e}^{(1+x+x^2)^{1/2}}$$

and finally

$$\begin{aligned} f'''(x) = {} & \frac{3}{8}\frac{(1+2x)^3}{(1+x+x^2)^{5/2}}\mathrm{e}^{(1+x+x^2)^{1/2}} - \frac{3}{8}\frac{(1+2x)^3}{(1+x+x^2)^2}\mathrm{e}^{(1+x+x^2)^{1/2}} \\ & - \frac{3}{2}\frac{(1+2x)}{(1+x+x^2)^{3/2}}\mathrm{e}^{(1+x+x^2)^{1/2}} + \frac{1}{8}\frac{(1+2x)^3}{(1+x+x^2)^{3/2}}\mathrm{e}^{(1+x+x^2)^{1/2}} \\ & + \frac{3}{2}\frac{(1+2x)}{(1+x+x^2)}\mathrm{e}^{(1+x+x^2)^{1/2}} \end{aligned}$$

Then compute $f(0) = \mathrm{e}$, $f'(0) = \mathrm{e}/2$, $f''(0) = \mathrm{e}$ and $f'''(0) = \mathrm{e}/8$ giving the Taylor polynomial

$$\mathrm{e} + \mathrm{e}x/2 + \mathrm{e}x^2/2 + \mathrm{e}x^3/48.$$

Note: Since this exercise requires careful administration and bulky computations, we have been lenient regarding minor errors

5. (a) The function is well-defined as long as the argument for the logarithm is positive. That is $\{(x, y) \mid y - x^2 > 0\}$. This is the region above, but not including, the parabola $y = x^2$.

   (b) We compute the stationary points.

$$\begin{cases} \frac{\partial}{\partial x}f(x,y) = \log(y-x^2) - \frac{2x^2}{y-x^2} = 0 \\ \frac{\partial}{\partial y}f(x,y) = \frac{x}{y-x^2} = 0 \end{cases}$$

   Then solving this system by first solving the second equation, giving $x = 0$, and then the first equation giving $y = 1$. So the only stationary point is $(0, 1)$. Then we need to investigate what the nature of the point is. Computing the second derivatives

$$\begin{cases} \frac{\partial^2}{\partial x^2}f(x,y) = -\frac{2x}{y-x^2} - \frac{4x^3}{(y-x^2)^2} - \frac{4x}{y-x^2} \\ \frac{\partial^2}{\partial y^2}f(x,y) = -\frac{x}{(y-x^2)^2} \end{cases}$$

But both $A = \frac{\partial^2}{\partial x^2} f(0,1)$ and $B = \frac{\partial^2}{\partial y^2} f(0,1) = 0$, so we can not draw a conclusion yet. We compute further

$$\frac{\partial^2}{\partial y \partial x} f(x,y) = \frac{1}{y - x^2} + \frac{2x^2}{y - x^2}$$

Evaluating this at $(0,1)$ gives that $C = \frac{\partial^2}{\partial y \partial x} f(0,1) = 2$. But then the point is a saddle point since $AB - C^2 < 0$.

6. If the line has to go through $(0,1)$ and $(1,0)$ then the slope must be -1. Hence $y'(0) = -1$. We compute the derivative by implicit differentiation

$$3y'y^2 + bxy' + by = 0$$

hence

$$y' = -\frac{by}{3y^2 + bx}.$$

Now at $(0,1)$ this becomes

$$y'(0) = -\frac{b}{3}$$

hence $b = 3$.

7. This is geometric series, hence it converges if and only if

$$\frac{1}{2b^2 - 1} < 1$$

Hence it converges for

$$2b^2 - 1 > 1$$

which means

$$b > 1 \text{ or } b < -1$$

To compute it at $b = 7$, it is important to realize that the series starts at $k = 1$. So the answer for general b is

$$\frac{\frac{1}{2b^2-1}}{1 - \frac{1}{2b^2-1}}$$

Now inserting $b = 2$ gives

$$\frac{\frac{1}{7}}{1 - \frac{1}{7}} = \frac{1}{6}.$$