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Sketch of the solutions to the examination paper in Mathematics for Economic and Statistical Analysis, Master Program, August 20, 2013

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1. This geometric series with $r = \dfrac{1}{1+x}$ is convergent if $-1 < r < 1$, i.e. only if $-1 < \dfrac{1}{1+x} < 1$. Consider two cases: (a): $1 + x < 0$ and (b) $1 + x > 0$. Multiplying both sides by $1 + x$ gives us in the case (a): $-1 - x > 1 > 1 + x$, which implies $x < -2$, and in the case (b): $-1 - x < 1 < 1 + x$, which implies $x > 0$. Hence the answer is $x < -2$ or $x > 0$.

2. a) $\displaystyle\int_1^e \frac{1-\ln x}{x^2}\,dx = \int_1^e \frac{1}{x^2}\,dx - \int_1^e \frac{\ln x}{x^2}\,dx = -\frac{1}{x}\Big|_1^e - \int_1^e \ln x \cdot \frac{1}{x^2}\,dx =$ ... integration by parts $\displaystyle ... = 1 - \frac{1}{e} - \Big(\ln x \cdot (-\frac{1}{x})\Big|_1^e - \int_1^e \frac{1}{x} \cdot (-\frac{1}{x})\,dx\Big) = 1 - \frac{1}{e} + \frac{1}{e} - \int_1^e \frac{1}{x^2}\,dx = 1 + \frac{1}{x}\Big|_1^e = \frac{1}{e}.$

b) $\displaystyle\int \frac{1}{t\ln t}\,dt =$ ... substitution $s = \ln t$, $ds = \dfrac{1}{t}\,dt$ gives $\displaystyle ... = \int \frac{1}{s}\,ds = \ln|s| + C = \ln|\ln|t|| + C.$

3. Inserting $x = 1$ into $e^{x^2+y(x)} + y(x)\ln x = x^2$ gives $e^{1+y(1)} = 1$. Hence $1 + y(1) = 0$, i.e. $y(1) = -1$.
The differentiation of the expression gives $e^{x^2+y}(2x+y') + y'\ln x + \dfrac{y}{x} = 2x$. Inserting $x = 1$ and $y = y(1) = -1$ implies $2 + y' - 1 = 2$, i.e. $y'(1) = 1$.
The second derivative of the expression is $e^{x^2+y}(2x+y')^2 + e^{x^2+y}(2+y'') + y''\ln x + \dfrac{y'}{x} + \dfrac{y'x - y}{x^2} = 2$. This time the insertion of $x = 1$, $y = y(1) = -1$ and $y' = y'(1) = 1$ gives $9 + 2 + y'' + 1 + 2 = 2$ from which we get $y'' = y''(1) = -12 < 0$. Hence the function is concave at the point $x = 1$.

4. The determinant of the coefficient matrix equals $7(16 - c^2)$. It equals 0 if and only if $c = \pm 4$. For all other values of c the system has an unique solution.

For $c = 4$ we consider the system $\left(\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 3 & -1 & 5 & 2 \\ 4 & 1 & 2 & 6 \end{array}\right)$. This may be reduced to $\left(\begin{array}{ccc|c} 1 & 0 & 1 & 8/7 \\ 0 & 1 & -2 & 10/7 \end{array}\right)$, which apparently has infinitely many solutions.

For $c = -4$ we have the system $\left(\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 3 & -1 & 5 & 2 \\ 4 & 1 & 2 & -2 \end{array}\right)$. This may be reduced to $\left(\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & -7 & 14 & -18 \end{array}\right)$, which obviously has no solutions.

5. The function is well defined and continuos for all real x except $x = 0$. The derivative, $f'(x) = \dfrac{e^x(x-1)}{x^2}$, is 0 if and only if $x = 1$. The second derivative is $f''(x) = e^x \cdot \dfrac{x^2 - 2x + 2}{x^3}$. Since $f''(1) = e > 0$ then $x = 1$ is a local minimum and $f(1) = e$.
If we check the sign of the first derivative we find that is is negative when $x < 0$ and for $0 < x < 1$, while it is positive when $x > 1$. Thus the function is decreasing for $x < 0$ and $0 < x < 1$, while increasing for $x > 1$. The second derivative is never zero ($x^2 - 2x + 2 = 0$ has no real solutions) and is negative for $x < 0$ and positive when $x > 0$. Thus the function is concave for $x < 0$ and convex for $x > 0$.
Finally, $\lim\limits_{x\to-\infty} f(x) = 0^-$, $\lim\limits_{x\to 0^-} f(x) = -\infty$, $\lim\limits_{x\to 0^+} f(x) = \infty$ and $\lim\limits_{x\to\infty} f(x) = \infty$. The function has no global extremum.

6. Partial derivatives are $f'_x = 6x + 3y$ and $f'_y = 3x + 2y + 3y^2$. In order to find the stationary points we solve the system of equations $6x + 3y = 0$ and $3x + 2y + 3y^2 = 0$. We find two solutions $P_1 = (0, 0)$ and $P_2 = (\frac{1}{12}, -\frac{1}{6})$

The second partial derivatives are $A = f''_{xx} = 6$, $B = f''_{xy} = 3$ and $C = f''_{yy} = 2 + 6y$. We study now the points in order:

$(0, 0)$: $A = 6$, $B = 3$, $C = 2$ and $AC - B^2 = 3$. A minimum point.

$(\frac{1}{12}, -\frac{1}{6})$: $A = 6$, $B = 3$, $C = 1$ and $AC - B^2 = -3$. A saddle-point.

7. Since $f'_x = (2 + 2x + y)e^{x-\frac{1}{2}y}$, $f'_y = (1 - x - \frac{1}{2}y)e^{x-\frac{1}{2}y}$, $f''_{xx} = (4 + 2x + y)e^{x-\frac{1}{2}y}$ and $f''_{yy} = (-1 + \frac{1}{2}x + \frac{1}{4}y)e^{x-\frac{1}{2}y}$ then the equality $f''_{xx} - 4f''_{yy} = a(f'_x + 2f'_y)$ reduces to $(4+2x+y)e^{x-\frac{1}{2}y} - 4\cdot(-1+\frac{1}{2}x+\frac{1}{4}y)e^{x-\frac{1}{2}y} = a((2 + 2x + y)e^{x-\frac{1}{2}y} + 2(1 - x - \frac{1}{2}y)e^{x-\frac{1}{2}y})$, i.e. $8e^{x-\frac{1}{2}y} = a(4e^{x-\frac{1}{2}y})$. Hence $a = 2$.

Paul