3-COL

Input: A graph $G=(V,E)$

Q: Is G 3-colorable? That is, is there a coloring $\sigma : V \to \{1,2,3\}$ s.t. $\{u,v\} \in E \Rightarrow \sigma(u) \neq \sigma(v)$

We've seen that 3-COL is NP-complete, as well as k-coloring:

k-COL

Input: A graph $G=(V,E)$, integer k

Q: Is G k-colorable? That is, is there a coloring $\sigma : V \to \{1,2,\ldots,k\}$ s.t. $\{u,v\} \in E \Rightarrow \sigma(u) \neq \sigma(v)$

For practice, we'll show 6-COL is NP-complete.

Step 1: 6-COL $\in$ NP

We show a 6-coloring can be verified in poly. time.

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check-coloring(G, σ)
  for v ∈ V(G)
    for neighbors u of v:
      if σ(u) = σ(v): return false     ] |N(v)|, size of neighborhood.
  if σ uses ≤ 6 colors: return true
  else return false
```

Runtime is polynomial, $O(|V|^2)$. [In fact, $O(|V|+|E|)$ achievable]

Therefore 6-COL lies in NP.

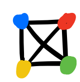
<u>Step 2</u>: 6-COL is NP-hard

We will reduce 3-COL $\leq_p$ 6-COL.

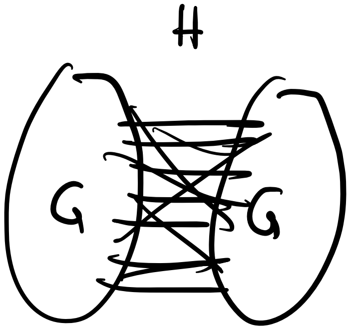
Let G be an input to the 3-COL problem.
We need to construct graph H to use as input for 6-COL such that

(a) H can be constructed in polynomial time, and

(b) $\underbrace{\text{G has a 3-coloring}}_{\text{G is a YES-instance}} \Leftrightarrow \underbrace{\text{H has a 6-coloring}}_{\text{H is a YES-instance}}$

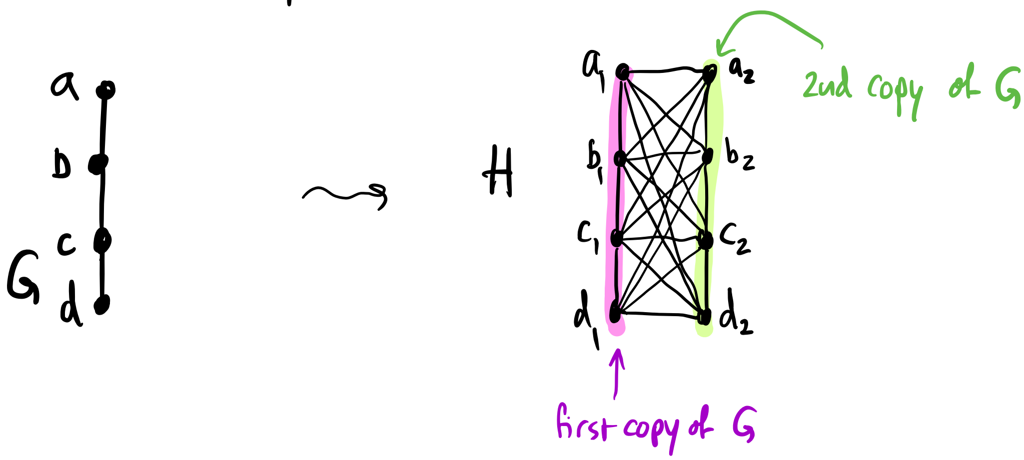
NB: $G=H$ does not work, (b) not satisfied. For example $G=H=K_4$ has a 6-coloring but no 3-coloring: 

From G we construct H by taking to copies of G and adding all possible edges between them.



This is known as the join of two graphs.

If v is a vertex of G, v_1 & v_2 denote the two copied vertices in H. Ex:



Can we construct H in polynomial time? sure, a sketch (fill in details as exercise):

1. For all $v \in V(G)$, add vertices v_1 & v_2
2. For all $\{u,v\} \in E(G)$, add edges $\{u_1, v_1\}$ & $\{u_2, v_2\}$
3. For all $u, v \in V(G)$ add edges $\{u_1, v_2\}$ & $\{u_2, v_1\}$

Runtime $O(|V(G)|^2)$ certainly possible, so (a) ok.

Now we show equivalence in (b).

$\Rightarrow$)

Sps G is a 3-colorable graph.

Let $\sigma : V \to \{1,2,3\}$ be such a coloring.

We define

$$\pi : V(H) \to \mathbb{N} \qquad \text{by}$$

$$\pi(v_i) = \begin{cases} \sigma(v) & \text{if } i=1 \\ \sigma(v)+3 & \text{if } i=2 \end{cases} \qquad \text{for all } v_i \in V(H)$$

(remember, each vertex of H is v_1 or v_2 for some $v \in V(G)$.)

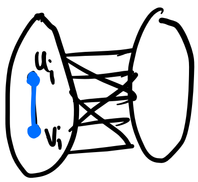
We need to show π is a 6-coloring of H.

The image of π is a (subset) of $\{1,2,3,\dots 6\}$ since codomain is $\{1,2,3\}$.

Just make sure no edge of H have same color on two end points.

Consider edge $\{v_i, u_j\}$ of H. Then $i,j \in \{1,2\}$ and $\{u,v\} \in E(G)$.

if $i=j=1$ then u_i, v_i in same copy of G, so $\{u,v\} \in E(G)$



and $\pi(u_i) = \sigma(u)$
$\pi(v_i) = \sigma(v)$

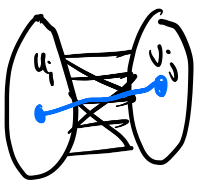
Since $\{u,v\} \in E(G)$ & σ a coloring have

$$\pi(u_1) = \sigma(u) \neq \sigma(v) = \pi(v_1)$$

Similar if $i=j=2$

If $i=1 \neq 2=j$ then u_i & v_j in different copies

and thus



$\pi(u_i) = \sigma(u) \in \{1,2,3\}$
$\pi(v_j) = \sigma(v) + 3 \in \{4,5,6\}$

so $\pi(u_i) \neq \pi(v_j)$

similar if $i=2 \neq 1=j$.

This shows

$$G \text{ 3-colorable} \Rightarrow H \text{ 6-colorable}$$

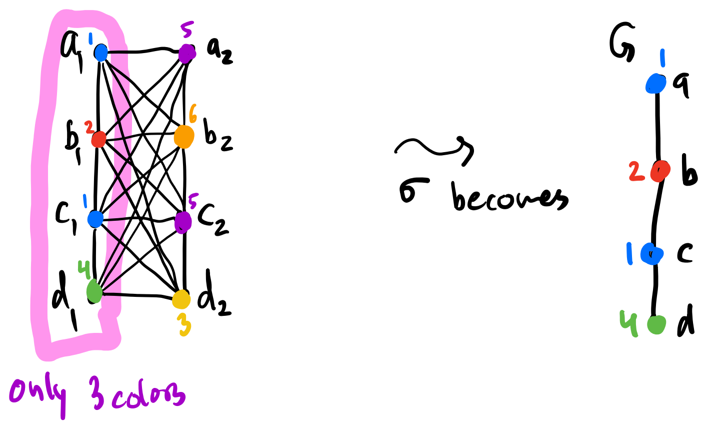
$\Leftarrow$) Say H is 6-colorable, let

$\pi : V(H) \to \{1,2,\dots,6\}$ be such coloring.

If π only uses ≤ 3 colors on the vertices of H with index 1, we can transfer that to a 3-coloring of G:

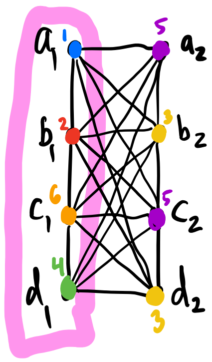
$$\sigma : V(G) \to \Upsilon \subseteq \{1,2,\dots,6\} \quad \text{for } |\Upsilon| \le 3$$

defined by $\sigma(v) = \pi(v_1) \ \forall v \in V(G)$.



If π uses > 3 colors on the vertices of H with index 1, I claim it uses < 3 colors on the vertices with index 2.

Why? Well, every u_1 is adjacent to every v_2 in H



> 3 colors

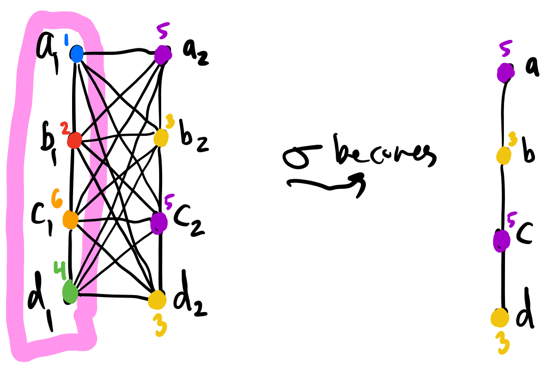
For $u, v \in V(G)$, so no color of vertex with index 1 can repeat on a vertex with index 2.

If $k > 3$ colors on the vertices with index 1, then at most $6 - k < 3$ colors on vertices with index 2.

Therefore, the coloring

$$\sigma : V(G) \to \Upsilon \subseteq \{1, 2, \ldots, 6\}$$

defined by $\sigma(v) = \pi(v_2)$ uses ≤ 3 colors:



> 3 colors

We've shown

$$H \text{ 6-colorable} \implies G \text{ 3-colorable}.$$

Therefore (b) also holds. We've shown 6-COL is NP-complete.