Largest consecutive sublist sum

input: list $L = [x_1, x_2, \dots, x_n]$

output: largest possible sum of a consecutive sublist of L

Ex 1 $\quad L = [\underbrace{1, 2, 3, 4}_{\text{whole list}}] \to 10$

Ex 2 $\quad L = [-1, 1, -1, 1, -1] \to 1$

many sublists with sum $= 1$

Ex 3 $\quad L = [\underbrace{1, 2, 3}_{6}, -10, 5, -10]$

Ex 3 shows greedy approach (pick max element and try to extend left and/or right) is not a good strategy.

For $i \le j$ define $S_{ij} = x_i + x_{i+1} + \dots + x_j$

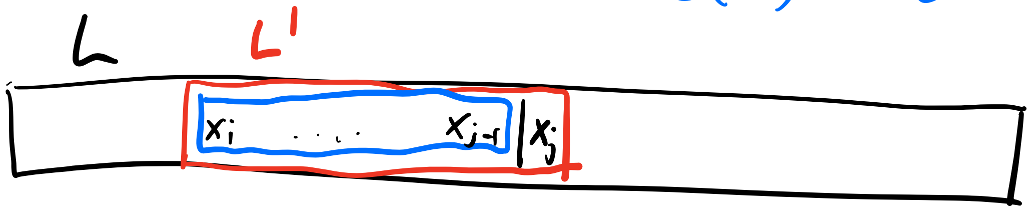
Solution to problem is rephrased as

$$\max \{ S_{ij} \mid 1 \le i \le j \le n \} \qquad (*)$$

Brute force: compute S_{ij} $\forall\, i \le j$.

for $j = 1, 2, \dots n$:
  for $i = 1, 2, \dots, j$:
    $S_{ij} \leftarrow x_i + \dots + x_j$

$1 + 2 + \dots + n$ sums $= \frac{n(n+1)}{2} \in O(n^2)$ with $j - i + 1 \le n$ elts each so maybe $O(n^3)$ or $O(n^2)$ runtime,



Say L' is consecutive sublist $L[i \dots j]$ with max sum. Then either $L' = [x_j]$ or the sublist $[x_i \dots x_{j-1}]$ must be best sublist in $L[1 \dots (j-1)]$.

Want to "reuse" computations as much as possible.

Define

$$\delta_j = \max\{S_{ij} \mid 1 \le i \le j\}$$

= max sum of consecutive sublist ending in j:th index of L.

Clearly

$$(*) = \max\{\delta_j \mid 1 \le j \le n\}$$

solves main problem.

Let's calculate δ_j "cleverly".

<u>Claim</u>

$$\delta_j = \begin{cases} x_1 & \text{if } j=1 \\ \max(x_j, x_j + \delta_{j-1}) & \text{otherwise.} \end{cases} \qquad (**)$$

Proof of claim

By induction on j.

Base $j=1$: Best sublist of $[x_1]$ is all of it, with sum x_1.

(IH) Sps (**) holds for some $j \geq 1$.

Consider δ_{j+1}. By definition $j+1 \geq 2$ and

$$\delta_{j+1} = \max \{ S_{i,j+1} \mid 1 \leq i \leq j+1 \}$$

// every sum $i<j+1$ $S_{i,j+1}$ ends with term x_{j+1} i.e. $S_{i,j+1} = S_{ij} + x_{j+1}$

$$= \max \left(S_{j+1,j+1}, \max \{ S_{i,j} + x_{j+1} \mid 1 \leq i \leq j \} \right)$$

$$= \max \left(x_{j+1}, \max \{ S_{ij} \mid 1 \leq i \leq j \} + x_{j+1} \right)$$ // apply IH

$$= \max \left(x_{j+1}, \delta_j + x_{j+1} \right)$$

This proves the claim.

Can now transform to algorithm

```
ALG(L: list of length n)
    D ← new list of length n      // will contain δ_i's
    D[1] ← L[1]                   // δ_1 = x_1
    for i = 2, 3, ..., n:
        D[i] ← max(L[i], L[i] + D[i-1])
    return max(D)
```

Correctness: almost immediate from claim, just note that $D[i] = \delta_i \ \forall\ 1 \le i \le n$ (inductively, if you want). Hence the output $\max(D)$ is

$$\max(\delta_1, \delta_2, \dots, \delta_n) = \max_j \{ S_{ij} \mid 1 \le i \le j \}$$

$$= \max \{ S_{ij} \mid 1 \le i \le j \le n \}.$$

which is what we want

Runtime: lines 1 & 4 (once) $O(1)$

lines 2, 5 & loop total $O(n)$

$\rightarrow$ together $O(n)$.

Recap matroids

$(E, \mathcal{F})$ a matroid if

(M1) $\mathcal{F} \neq \emptyset$

(M2) $X \subseteq Y$ & $Y \in \mathcal{F} \Rightarrow X \in \mathcal{F}$

(M3) $X, Y \in \mathcal{F}$ & $|X| < |Y| \Rightarrow \exists y \in Y \setminus X$ s.t. $X \cup y \in \mathcal{F}$

An $\subseteq$-maximal element in $\mathcal{F}$ is a basis.

Thm (lecture) Bases of matroid M have same size.

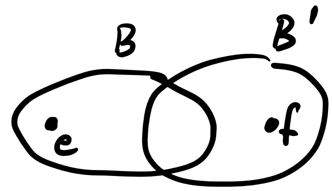
Show "Basis exchange property":

$M=(E,\mathcal{F})$ a matroid with two bases B & B'.

If $b \in B\setminus B'$ then $\exists\, b' \in B'\setminus B$ s.t.

$$(B-b)\cup b' \text{ is a basis of } M.$$

Pf Say $e \in B\setminus B'$.



Then $\left.\begin{array}{r} B-e \subseteq B \\ B\in\mathcal{F} \end{array}\right\} \overset{M2}{\Longrightarrow} B-e\in\mathcal{F}.$

Now $\underbrace{|B-e|}_{\in\mathcal{F}} = |B|-1 \underset{\text{thm}}{=} |B'|-1 < \underbrace{|B'|}_{\in\mathcal{F}}$ and

(M3) implies $\exists\, f\in B'\setminus(B-e)$ s.t.

$$(B-e)\cup f \in \mathcal{F}.$$

In fact, $e\in B\setminus B'$ means $f\in B'\setminus(B-e) = B'\setminus B$.

$(B-e)\cup f$ is independent (= in $\mathcal{F}$). Is it a basis?

Yes! If not, $(B-e)\cup f \subsetneq C$ for basis C

and $|C| > |(B-e)\cup f| = |B|$, contradiction to B being a basis. $\square$

__For you__

Let $M=(E,\mathcal{F})$ be a matroid & $w: E \to \mathbb{R}_{\geq 0}$ a weighting function. Show

$w(e) \neq w(f)$ $\forall e,f \in E$ $\Rightarrow$ An optimal* basis of M (wrt w) is unique.

*B is optimal basis wrt w if $w(B) \geq w(B')$ $\forall$ bases B' of M

Hint: Assume, for contradiction, two optimal bases B & B'. Pick element in symmetric difference $B \Delta B'$ () of minimal weight. Then basis exchange property & find a contradiction.