Let Π be an optimization problem, usually NP-hard.

We'll recap what it means for Π to:

- admit a PTAS and FPTAS

  (Fully) Polynomial-Time Approximation Scheme

- be FPT

  Fixed Parameter Tractable

Def <u>Approximation Scheme</u> is an algorithm

Input: instance I (so, the input) of Π

& $\varepsilon > 0$

output: an $(1+\varepsilon)$-approximation i.e.

$$\frac{OPT(I)}{1+\varepsilon} \leq ALG(I,\varepsilon) \leq (1+\varepsilon)\,OPT(I)$$

A PTAS is an approx. scheme whose runtime is polynomial in $|I|$

An FPTAS is an approx. scheme whose runtime is polynomial in $|I|$ <u>and ε</u>

Practice

Say $f(n, \varepsilon)$ is the runtime of an approx. scheme. For the following, determine if the algorithm is a PTAS, an FPTAS or neither:

(a) $f \in O(n^2 + \frac{1}{\varepsilon})$

(b) $f \in O(n^{2+1/\varepsilon})$

(c) $f \in O(2^{n+1/\varepsilon})$

(d) $f \in O((\frac{1}{\varepsilon})^{\sqrt{n}})$

(e) $f \in O((\log(n))^{1/\varepsilon})$

Solution neither: (c) (d) PTAS: (b) (e) FPTAS: (a).

Polynomial in n: treat ε as constant. Is it bounded by some polynomial $p(n)$?

Polynomial in ε: treat n as constant. Is it bounded by some polynomial $p(\varepsilon)$?

You saw a PTAS for simple knapsack in the lectures. Subset Sum, from last tutorial, also admits a PTAS.

Some problems, like vertex cover and max 3-SAT, are unlikely to admit PTAS: there exist results of the type:

if there exists an $\left(\frac{7}{8}-\delta\right)$-approximation algorithm of max-3SAT for some $0<\delta<1$, then $P=NP$.

What about FPT?

Def

A problem Π is fixed-parameter tractable for some parameter k

if there is an algorithm with:

Input: instance I (so, the input) of Π, sometimes k

Output: optimal/correct answer for I

and runtime $O(f(k) \cdot p(|I|))$ for a polynomial p and a function f.

- p not allowed to depend on k
- f not allowed to depend on $|I|$

If k is constant, runtime is $O(p(|I|))$ – polynomial.

Practice

Could any of these functions be runtime f of an FPT-alg? Here $n = |I|$ & k is the parameter

(a) $f \in O(2^{k+n})$ no

(b) $f \in O(2^k \cdot n)$ yes

(c) $f \in O(kn)$ yes

$\in O(k^2 + n^2)$

(d) $O(3^k n) \in O(3^{2k} + n^2)$ yes

(e) $O(3^k n^{\log k})$ no

trick: $0 \leq (x-y)^2 = x^2 - 2xy + y^2$

$\Rightarrow 2xy \leq x^2 + y^2$

3-Hitting set

Input: A collection $\mathcal{C} = \{S_1, S_2, \ldots, S_n\}$ of sets with 3 elements each (can think of $S_i \subseteq \{1, 2, \ldots, m\}$) & integer k

Question: is there a $\underbrace{\text{hitting set } H}_{H \cap S_i \neq \emptyset \ \forall i}$ s.t. $|H| \leq k$?

Ex

$S_1 : \{1, 2, 3\}$

$S_2 : \{3, 4, 5\}$

$S_3 : \{5, 6, 7\}$

$S_4 : \{8, 9, 10\}$

$\{1, 3, 5, 8\}$ is a hitting set

$\{1, 3, 5\}$ is not

$\{1, 5, 8\}$ is a hitting set

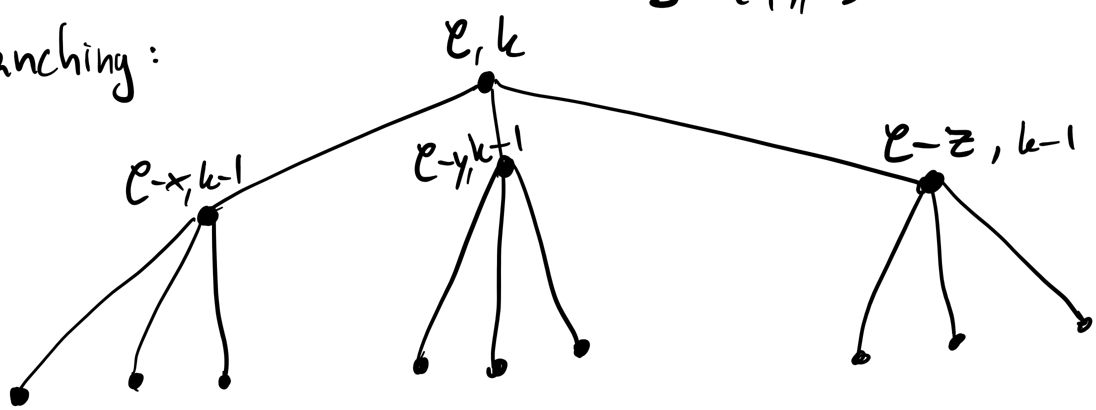
Not explicitly about but many connections (can discuss later). Slightly informed algorithm:

ALG($\mathcal{C}$, k)

1. If $\mathcal{C} = \emptyset$: return YES — $O(1)$
2. If $k = 0$: return NO — $O(1)$
3. Find any $S \in \mathcal{C}$ — $O(1)$
4. For each $x \in S$: — 3 repeats
5. | If ALG($\mathcal{C}$ − {sets containing x} $k-1$) returns YES, then return YES
6. Return NO

$S = \{x, y, z\}$

Branching:



Worst case when $\mathcal{C}$ only reduced by one set at a time.

The branching vector here is $(1,1,1)$

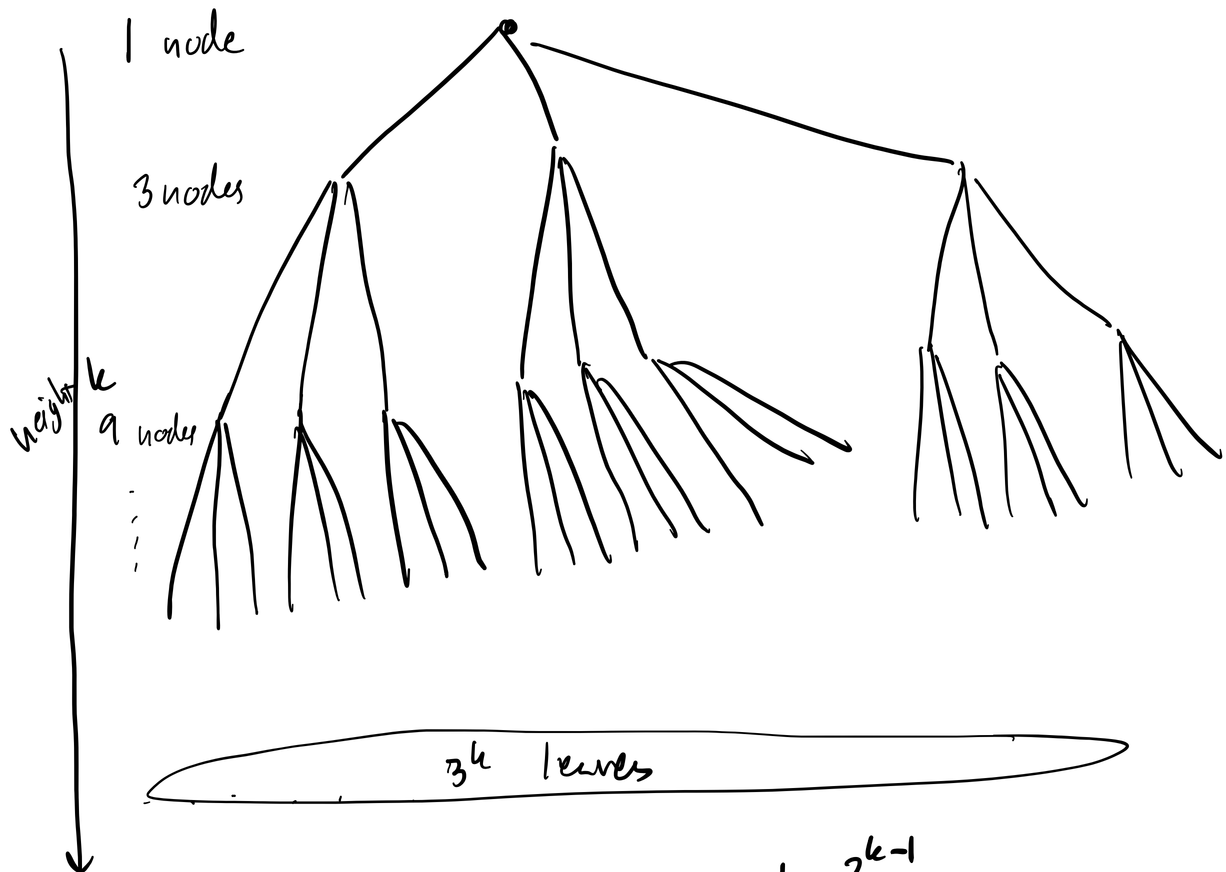
$k_1 = k_2 = k_3 = 1 \quad p = 3$

By recipe: solve

$$k = \max k_i \quad \alpha^1 - \sum_{i=1}^{p} \alpha^{(1-1)} = \alpha - 3 = 0 \Leftrightarrow \alpha = 3$$

So runtime is $O(3^k)$.

Easy in this case:



$$1 + 3 + 9 + \dots + 3^k = \frac{1 - 3^{k-1}}{1 - 3} = \frac{1}{2}(3^{k-1} - 1) \in O(3^k)$$

What would the runtime be with a branching vector $(1, 1, 6)$?

$k = \max(1,1,6) = 6$

$$\alpha^6 - \sum_{i=1}^{3} \alpha^{6-k_i} = \alpha^6 - (\alpha^5 + \alpha^5 + 1) = 0$$

$$\Leftrightarrow \quad \alpha^6 - 2\alpha^5 - 1 = 0 \quad \Leftrightarrow \quad \begin{array}{l} \alpha = -0{,}8812\ldots \\ \text{or} \\ \alpha = 1.2852\ldots \end{array}$$