1 Introduction

This is a collection of selected lemmas, propositions and theorems in topology from the book Introduction to Topological Manifolds by John M. Lee[1]. Refer to this book for precise wording and definitions.
Mouse over underlined text for a pop-up definition.
Some definitions that are not enumerated in the book have been included, these have been enumerated as **2.3X** to indicate roughly where to find them in the textbook.

2 Topological Spaces

Topologies

2.8 Proposition. Let X be a topological space (A set X with a topology $\mathscr{T}$) and let $A \subseteq X$ be any subset. Then

a) A point is in Int(A) $\Leftrightarrow$ it has a neighborhood contained in A.

b) A point is in Ext (A) $\Leftrightarrow$ it has a neighborhood contained in $X \setminus A$

c) A point is in $\partial(A)$ $\Leftrightarrow$ each of its neighborhood s contains both a point in A and a point in $X \setminus A$.

d) A point is in the closure of A, denoted $\overline{A}$ $\Leftrightarrow$ each of its neighborhood s contains a point of A.

e) $\overline{A} = A \cup \partial A = \text{Int}(A) \cup \partial(A)$

f) Int(A) and Ext (A) are open in X. The closure of A and ∂A are closed in X.

g) The following are equivalent:

   i) A is open in X.

   ii) $A =$Int$A$

   iii) A contains none of its boundary points.

   iv) Every point of A has a neighborhood contained in A.

h) The following are equivalent:

   i) A is closed in X.

   ii) $A = \overline{A}$

   iii) A contains all of its boundary points.

   iv) Every point of X\A has a neighborhood contained in $X \setminus A$.

Definition. closed set
Definition. dense set

Convergence and Continuity

2.15 Proposition. A map between topological spaces is continuous if and only if the preimage of every closed subset is closed.

2.17 Proposition. Let X, Y and Z be topological spaces.

a) Every constant map $f : X \to Y$ is continuous .

b) The identity map $\mathrm{Id}_X : X \to X$ is continuous .

c) If $f : X \to Y$ is continuous , so is the restriction of f onto any open subset of X.

d) If $f : X \to Y$ and $g : Y \to Z$ are both continuous , then so is their composition $g \circ f : X \to Z$

2.19 Proposition. (Local Criterion for Continuity) A map $f : X \to Y$ between topological spaces is continuous if and only if each point of X has a neighborhood on which (the restriction of) f is continuous .

2.30 Proposition. Suppose X and Y are topological space, and $f : X \to Y$ is any map.

a) f is continuous $\Leftrightarrow f(\overline{A}) \subseteq \overline{F(A)}$ for all $A \subseteq X$.

b) f is closed $\Leftrightarrow f(\overline{A}) \supseteq \overline{f(A)}$ for all $A \subseteq X$.

c) f is continuous $\Leftrightarrow f^{-1}(\mathrm{Int} B) \subseteq \mathrm{Int}(f^{-1}(B))$ for all $B \subseteq Y$.

d) f is open $\Leftrightarrow f^{-1}(\mathrm{Int} B) \supseteq \mathrm{Int}(f^{-1}(B))$ for all $B \subseteq Y$.

2.31 Proposition (Properties of Local Homeomorphisms).

a) Every homeomorphism is a local homeomorphism .

b) Every local homeomorphism is continuous and open .

c) Every bijective local homeomorphism is a homeomorphism .

Hausdorff Spaces

2.37 Proposition. Let X be a Hausdorff space.

a) Every finite subset of X is closed .

b) If a sequence (p_i) in X converges to a limit $p \in X$, the limit is unique.

2.39 Suppose X is a Hausdorff space and $A \subseteq X$. If $p \in X$ is a limit point of A, then every neighborhood of p contains infinitely many points of A.

2.3X Definition: Basis A collection $\mathscr{B}$ of subsets of X is called a *basis for the topology of* X is the following conditions hold

i) Every element of $\mathscr{B}$ is an open subset of X.

ii) Every open subset of X is the union of some collection of elements of $\mathscr{B}$.

Bases and Countability

2.43 Proposition. Let X and Y be topological spaces and let $\mathscr{B}$ be a basis for Y. A map $f : X \to Y$ is continuous if and only if for every basis subset $B \in \mathscr{B}$, the subset $f^{-1}(B)$ is open in X.

2.44 Proposition. Let X be a set, and suppose $\mathscr{B}$ is a collection of subsets of X. Then $\mathscr{B}$ is a basis for some topology on X is and only if it satisfies the following conditions:

i) $\cup_{B \in \mathscr{B}} B = X$ and

ii) if $B_1, B_2 \in \mathscr{B}$ and $x \in B_1 \cap B_2$ then $\exists B_3 \in \mathscr{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

If so, there is a *unique* topology on X for which $\mathscr{B}$ is a basis , called the *topology generated by* $\mathscr{B}$.

Countability Properties

Four different countability properties, first countable , second countable , separable and Lindelöf .

2.47 Lemma (Nested Neighborhood Basis Lemma). Let X be a first countable space. For every $p \in X$, there exists a nested neighborhood basis

2.48 Lemma (Sequence Lemma). Suppose X is a first countable space, A is any subset of X, and x is any point in X

a) $x \in \overline{A} \Leftrightarrow x$ is a limit point of a sequence of points in A.

b) $x \in$Int$A \Leftrightarrow$ every sequence in X converging to x is eventually in A.

c) A is closed $\Leftrightarrow A$ contains every limit of every convergent sequence of points in A.

d) A is open in $X \Leftrightarrow$ every sequence in X converging to a point of A is eventually in A.

2.50 Theorem (Properties of Second Countable Spaces). Suppose X is a second countable space.

a) X is first countable .

b) X contains a countable dense subset.

c) Every open cover of X has a countable subcover .

2.5X Definition. separable space.

2.5X Definition. Lindelöf space.

Manifolds

2.5X Definition. locally Euclidean

2.52 Lemma. A topological space M is locally Euclidean of dimension n if either of the following properties hold:

a) Every point of M has a neighborhood homeomorphic to an open ball in $\mathbb{R}^n$

b) Every point of M has a neighborhood homeomorphic to $\mathbb{R}^n$.

2.5X Definition. Manifold. An n-dimensional topological manifold is a second countable Hausdorff space that is locally Euclidean of dimension n.

2.53 Proposition. Every open subset of an n-manifold is an n-manifold .

2.55 Theorem (Invariance of Dimension). If $m \neq n$, a nonempty topological space cannot be both an m-manifold and an n-manifold .

2.56 Proposition. A separable metric space that is locally Euclidean of dimension n is an n-manifold .

2.5X Definition. Closed n-dimensional upper half-space $\mathbb{H}^n \subseteq \mathbb{R}^n$

$$\mathbb{H}^n = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$$

2.5X Definitions.
$\partial\mathbb{H}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n = 0\}$
$\text{Int}\mathbb{H}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n > 0\}$.
(U, ψ) is a coordinate chart for M
(U, ψ) is an interior chart if $\psi(U)$ is an open subset of $\mathbb{R}^n$ (which includes the case in which $\psi(U)$ is an open subset of $\mathbb{H}^n$).
(U, ψ) is a boundary chart if $\psi(U)$ is an open subset of $\mathbb{H}^n$ with $\psi(U) \cap \partial\mathbb{H}^n \neq \emptyset$

2.58 Proposition. If M is an n-dimensional manifold with boundary , then Int$M$ is an open subset of M, which itself is an n-dimensional manifold without boundary .

2.59 Theorem (Invariance of the Boundary). If M is a manifold with boundary , then a point of M cannot be both a boundary point and an interior point. Thus ∂M and $\text{Int}\, M$ are disjoint subsets and $\partial M \cup \text{Int}\, M = M$.

2.60 Corollary. If M is a nonempty n-dimensional manifold with boundary , then ∂M is closed is M and M is an n-manifold if and only if $\partial M = \emptyset$.

3 New Spaces From Old

3.5 Proposition. Suppose S is a subspace topology of the topological space X .

a) If $U \subseteq S \subseteq X$, and S is open in X, then U is open in X. The same is true with "closed " instead of "open".

b) If U is a subset of S that is either open or closed in X, then it is also open or closed in S, respectively.

3.8 Theorem (Characteristic Property of the Subspace Topology Suppose X is a topological space and $S \subseteq X$ is a subspace. For any topological space Y, a map $f : Y \to S$ is continuous $\Leftrightarrow$ the composite map $\iota_S \circ f : Y \to X$ is continuous:

$$\begin{array}{ccc} & & X \\ & \overset{\iota_S \circ f}{\nearrow} & \uparrow \iota_S \\ Y & \xrightarrow[f]{} & S \end{array}$$

3.9 Corollary. If S is a subspace of the topological space X, the inclusion map $\iota_S : S \hookrightarrow X$ is continuous .

3.10 Corollary. Let X and Y be topological spaces, $f : X \to Y$ is continuous . Then

a) Restricting the Domain: The restriction of f to any subspace $S \subseteq X$ is continuous .

b) Restricting the Codomain: If T is a subspace of Y that contains $f(X)$, then $f : X \to T$ is continuous .

c) Expanding the Codomain: If Y is a subspace of Z, then $f : X \to Z$ is continuous .

3.11 Proposition. Suppose S is a subspace of the topological space X.

a) If $R \subseteq S$ is a subspace of S, then R is a subspace of X. I.o.w. the subspace topologies that R inherits from S and from X agree.

b) If $\mathscr{B}$ is a basis for the topology of X, then

$$\mathscr{B}_S = \{B \cap S : B \in \mathscr{B}\}$$

is a basis for the topology on S.

c) If (p_i) is a sequence of points in S and $p \in S$, then $p_i \to p \in S$ if and only if $p_i \to p \in X$.

d) Every subspace of a Hausdorff space if Hausdorff .

e) Every subspace of a first countable space is first countable .

f) Every subspace of a second countable space is second countable .

3.16 Proposition. A continuous injective map that is either open or closed is a topological embedding .

3.18 Proposition. A surjective topological embedding is a homeomorphism .

3.23 Lemma.(Gluing Lemma) Let X and Y be topological spaces, and let $\{A_i\}$ be either an arbitrary open cover of X, or a finite closed cover of X. Suppose that we are given continuous maps $f_i : A_i \to Y$ that agree on overlaps: $f_i|_{A_i \cap A_j} = f_j|_{A_i \cap A_j}$. Then there exists a unique continuous map $f : X \to Y$ whose restriction to each A_i is equal to f_i.

3.24 Theorem.(Uniqueness of the Subspace Topology) Suppose S is a subset of a topological space X. The subspace topology on S is the unique topology for which the characteristic property (**3.8**) holds.

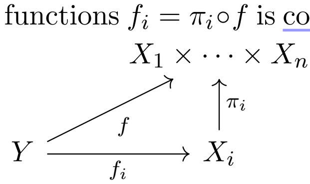
Product Spaces

Suppose $X_1, X_2, \ldots, X_n$ are arbitrary topological spaces. On their Cartesian product $X_1 \times \cdots \times X_n$, we define the product topology to be the topology generated by the basis

$$\mathscr{B} = \{U_1 \times \cdots \times U_n : U_i \text{ is an open subset of } X_i\}$$

The topological space is called a product space, the basis subsets of the form $U_i \times \cdots \times U_n$ is called product open subsets.

3.27 Theorem.(Characteristic Property of the Product Topology Suppose $x_1 \times \cdots \times X_n$ is a product space. For any topological space Y, a map $f_y \to X_1 \times \cdots \times X_n$ is continuous if and only if each of its component functions $f_i = \pi_i \circ f$ is continuous , where $\pi_i : X_1 \times \cdots \times X_n \to X_i$ is the canonical projection:



3.28 Corollary. If $x_1, \ldots, X_n$ are topological spaces, each canonical projection $\pi_i : X_1 \times \cdots \times X_n \to X_i$ is continuous .

3.30 Theorem.(Uniqueness of the Product Topology.) Let $X_1, \ldots, X_n$ be topological spaces. The product topology $X_1 \times \cdots \times X_n$ is the unique topology for which the characteristic property (**3.27**) holds.

3.31 Proposition. Let $X_1, \ldots, X_n$ be topological spaces.

a) The product topology is "associative" in the sense that three topologies on the set $X_1 \times X_2 \times X_3$, obtained by thinking of it as $X_1 \times X_2 \times X_3$, $(X_1 \times X_2) \times X_3$ or $X_1 \times (X_2 \times X_3)$ are all equal.

b) For any $i \in \{1, \dots, n\}$ and any point $x_j \in X_j$, $j \neq i$, the map $f : X_1 \to X_1 \times \cdots \times X_n$ given by

$$f(x) = (x_1, \dots, x_{i-1}, x, x_{i+1}, \dots x_n)$$

is a topological embedding of X_i onto the product space.

c) Each canonical projection $\pi_i : X_1 \times \cdots \times X_n \to X_i$ is an open map.

d) If for each i, $\mathscr{B}_i$ is a basis for the topology on X_i, then the set

$$\{B_1 \times \cdots \times B_n : B_i \in \mathscr{B}_i\}$$

is a basis for the product topology on $X_1 \times \cdots \times X_n$.

e)

References

[1] John M. Lee. *Introduction to Topological Manifolds*, volume 202 of *Graduate Texts in Mathematics*. Springer, New York, NY, 2 edition, 2011.