Topology, Fall 2025

Gregory Arone

Homework exercise #4

Due: Thursday Oct 30, 2025

Instructions: We strongly prefer that you typeset your answers with Latex or similar. Handwritten assignments will be graded only if easy to read.

You may work together on the assignment, but write your own solution and do not copy. You also are welcome to consult the instructors, after you made a decent effort on a problem. Tell us what you have tried, and we will be happy to provide some direction if needed.

Please, please, do not seek inappropriate help on the internet or from ChatGPT and the like.

1. Let X be a path-connected space, with basepoint x_0. We know that $\pi_1(X, x_0)$ can be identified with the set of pointed homotopy classes of maps $(S^1, s_0) \to (X, x_0)$. Let $[S^1, X]$ be the set of unpointed homotopy classes of maps from S^1 to X. Prove that $[S^1, X]$ can be identified with the set of conjugacy classes of $\pi_1(X, x_0)$. (20 points)

   Hint: Show that there is a natural function $\pi_1(X, x_0) \to [S^1, X]$. Prove that the function is surjective (this may seem slightly surprising at first glance, and here it is important that X is path-connected). Then analyse which elements of $\pi_1(X)$ are mapped to the same element of $[S^1, X]$.

2. Let $f \in \text{mor}_{\mathcal{C}}(X, Y)$ be a morphism in some category $\mathcal{C}$. We say that f is a *monomorphism* if for every pair of morphisms $\alpha, \beta \in \text{mor}_{\mathcal{C}}(A, X)$, if $\alpha \neq \beta$ then $f \circ \alpha \neq f \circ \beta$. Dually, we say that f is an *epimorphism* if for any every pair of morphisms $\gamma, \delta \in \text{mor}_{\mathcal{C}}(Y, Z)$, if $\gamma \neq \delta$ then $\gamma \circ f \neq \delta \circ f$. We say that f is a *split epimorphism* if there exists a morphism $g \in \text{mor}_{\mathcal{C}}(Y, X)$ such that $f \circ g$ is the identity on Y. Finally, f is an *isomorphism* if there exists a morphism $g \in \text{mor}_{\mathcal{C}}(Y, X)$ such that $g \circ f$ is the identity on X and $f \circ g$ is the identity on Y.

   (a) Prove that a split epimorphism is in fact an epimorphism. Show an example of an epimorphism in some category that is not split. (10 points)

   (b) Prove that if f is a monomorphism and a split epimorphism, then it is an isomorphism. (Just FYI, there is a dual result involving a split monomorphism and an epimorphism, that is proved similarly.) (10 points)

   (c) Let $\underline{2}$ be a category with two objects, called a and b, and a single morphism $i \colon a \to b$, along with the identity morphisms on a and b. Is i a monomorphism? An epimorphism? An isomorphism? (10 points)