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Exam / Tentamen
Set Theory & Metamathematics
MM7048, 7,5 hp, VT2022
Mon, 2022-05-30

This exam consists of 3 questions, worth a total of 40 points. Note that some subproblems are intentionally quite challenging, and you do not need to complete all questions in order to achieve a good grade. You may submit your answers in either English or Swedish. Write clearly and motivate your answers carefully.

Good luck! — Lycka till!

1. **Restricted forms of choice in HFS.** Work in ZFU, with a countably infinite set of atoms A.

   (a) Show that a set $X \subseteq A$ has finite support if and only if it is either finite or cofinite.

   (b) Give an example of a set $X \subseteq \mathcal{P}(A)$ that has finite support, but is neither finite nor cofinite.

   For any set X, let $\mathrm{AC}[X]$ be the axiom of choice for subsets of X: that is, the statement "for any family $(Y_i)_{i \in I}$ of non-empty subsets of X, there exists some choice function $f \in \prod_i Y_i$".

   (c) Show that $\mathrm{AC}[\mathbb{R}]$ is equivalent to $(\mathrm{AC}[\mathbb{R}])^{\mathrm{HFS}}$.

   (d) Give a set $X \in \mathrm{HFS}$ such that $\mathrm{AC}[X]$ is not equivalent to $\mathrm{AC}[X]^{\mathrm{HFS}}$.

2. **Well-founded and ill-founded models of set theory.** Work in ZFC. Say that a structure $\mathcal{M} = (M, \varepsilon)$ for the language of set theory is *well-founded* just if ε is well-founded in the usual sense: that is, if every non-empty $X \subseteq M$ has some ε-minimal element, or equivalently, if there is no infinite $\ni$-descending sequence $x_0 \ni x_1 \ni x_2 \ni \ldots$ Conversely, call a structure *ill-founded* if it is not well-founded.

   (a) Show that any transitive standard structure $(N, \in)$ is well-founded.

   (b) Show that if $\mathcal{M} = (M, \varepsilon)$ is a well-founded model of ZF, then it is isomorphic to some transitive standard model $(N, \in)$.

   (c) Show that if $\mathcal{M}$ is a well-founded model of set theory, then $[\![(\omega, <)]\!]^{\mathcal{M}} \cong (\omega, <)$ — "$\mathcal{M}$ has standard arithmetic". (Here $[\![(\omega, <)]\!]^{\mathcal{M}}$ means the interpretation in $\mathcal{M}$ of the usual ZF-definition of ω, together with its usual ordering.)

   (d) Show that if ZF is consistent, then there exists some ill-founded model of ZF.

   (e) *Optional, not for credit: discuss why this doesn't contradict the fact that in any model of ZF, the foundation axiom certainly holds!*

3. **Equivalence of variant Gentzen systems.** In class, we considered the classical Gentzen system **G3**. Write $\mathbf{G3}_{\to}$ for its *implicational fragment*: that is, the system generated just by the axiom rule, and the rules for $\to$:

$$\frac{}{P, \Gamma \Rightarrow \Delta, P}\,\mathrm{Ax} \quad (P \text{ atomic})$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi \qquad \psi, \Gamma \Rightarrow \Delta}{\varphi \to \psi, \Gamma \Rightarrow \Delta}\,\mathrm{L}{\to} \qquad\qquad \frac{\varphi, \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \to \psi}\,\mathrm{R}{\to}$$

Simlarly, write $\mathbf{G1}_{\to}$ for the implicational fragment of the system that Troelstra and Schwichtenberg call **G1c** (we did not consider this in class); it consists of the same rules for $\to$, a slightly different axiom rule, and explicit rules for weakening and contraction:

$$\frac{\Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}\,\mathrm{LW} \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \varphi}\,\mathrm{RW} \qquad \frac{\varphi, \varphi, \Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}\,\mathrm{LC} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi, \varphi}{\Gamma \Rightarrow \Delta, \varphi}\,\mathrm{RC}$$

$$\frac{}{\varphi \Rightarrow \varphi}\,\mathrm{Ax}' \qquad \frac{\Gamma \Rightarrow \Delta, \varphi \qquad \psi, \Gamma \Rightarrow \Delta}{\varphi \to \psi, \Gamma \Rightarrow \Delta}\,\mathrm{L}{\to} \qquad \frac{\varphi, \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \to \psi}\,\mathrm{R}{\to}$$

Here in all rules, φ and ψ range over *purely implicational* formulas, i.e. generated just by atomic propositions and $\to$, using no other logical connectives.

(a) Show that the general axiom rule

$$\frac{}{\varphi, \Gamma \Rightarrow \Delta, \varphi}\,\mathrm{Ax}_{\mathrm{gen}}$$

is admissible over $\mathbf{G3}_{\to}$.

(b) Show that $\mathrm{Ax}_{\mathrm{gen}}$ is also admissible over $\mathbf{G1}_{\to}$.

(c) Show that the the weakening rules LW, RW are admissible over $\mathbf{G3}_{\to}$.

(d) Show that the contraction rules LC, RC are admissible over $\mathbf{G3}_{\to}$.

(e) Conclude that the systems $\mathbf{G3}_{\to}$ and $\mathbf{G1}_{\to}$ are equivalent.

(As usual in such proofs, don't feel obliged to be pedantically thorough: when several cases in a proof are similar, feel free to say so and give just as many cases as you feel illustrate the issues involved.)

——— End of exam — Slut på provet ———