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Re-exam / Omentamen
Set Theory & Metamathematics
MM7048, 7,5 hp, VT2022
Wed, 2022-08-10

This exam consists of 3 questions, worth a total of 40 points. Note that some subproblems are intentionally quite challenging, and you do not need to complete all questions in order to achieve a good grade. You may submit your answers in either English or Swedish. Write clearly and motivate your answers carefully.

Good luck! — Lycka till!

1. **Finiteness/infiniteness in permutation models.** Work in ZFU, with an countable set of atoms A. Consider HFS as we constructed it: the hereditarily finitely supported sets under the action of $\mathrm{Sym}(A)$.

   Recall that we defined a set to be *finite* if it admits a bijection to some $n \in \omega$, and infinite otherwise.

   Say a set x is *Dedekind-infinite* if there is some injection $f : x \to x$ that is not surjective, and *Dedekind-finite* otherwise.

   In some of the literature (especially older work), *Dedekind-(in)finite* is taken as the main definition of finiteness/infiniteness; the definition we've used may then be called *cardinal-(in)finite*, among other names. Under AC, they coincide; the goal of this problem is to show that without choice, they may diverge.

   (a) Show that any Dedekind-infinite set admits some injection from ω.

   (b) Assuming the axiom of choice, show that a set is infinite if and only if it is Dedekind-infinite.

   (c) Show that $\mathrm{HFS} \vDash$ "A is infinite". *(Note any absoluteness results that you use.)*

   (d) Show that $\mathrm{HFS} \vDash$ "there is no injection $\omega \to A$".

   It follows that $\mathrm{HFS} \vDash$ "A is Dedekind-finite, but not (cardinal-)finite", and hence that assuming "ZFU with a countable set of atoms" is consistent, ZFU does not prove that Dedekind-finiteness implies (cardinal-)finiteness.

2. **Cardinality calculations for the constructible universe.**

   Recall the "definable power set" construction $\mathcal{D}(X)$, used in the definition of $\mathbf{L}_{\alpha+1} \coloneqq \mathcal{D}(\mathbf{L}_\alpha)$.

   (a) Give the definition of $\mathcal{D}(X)$. *(We discussed a couple of different equivalent definitions of $\mathcal{D}$; any one of them is acceptable here.)*

   (b) Show that $|\mathcal{D}(X)| = |X|$ whenever $|X| \geq \aleph_0$.

   (c) Show that for each ordinal α, $\alpha \subseteq \mathbf{L}_\alpha$. *(Note any facts about absoluteness you use.)*

   (d) Show that for each ordinal $\alpha \geq \omega$, $|\mathbf{L}_\alpha| = |\alpha|$.

   (e) Show that assuming $\mathbf{V} = \mathbf{L}$, the $\mathbf{L}$-rank $\mathrm{rk}_{\mathbf{L}}(x)$ of a set may differ from its ordinary rank $\mathrm{rk}(x)$.

3. **Failures of admissibility in Gentzen systems.**

In class, we considered the classical Gentzen system **G3**. Write $\mathbf{G3}_{\rightarrow}$ for its *implicational fragment*: that is, the system generated just by the axiom rule, and the rules for $\rightarrow$:

$$\frac{}{P, \Gamma \Rightarrow \Delta, P}\ \mathrm{Ax} \quad (P \text{ atomic})$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi \qquad \psi, \Gamma \Rightarrow \Delta}{\varphi \rightarrow \psi, \Gamma \Rightarrow \Delta}\ \mathrm{L}{\rightarrow} \qquad\qquad \frac{\varphi, \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \rightarrow \psi}\ \mathrm{R}{\rightarrow}$$

We will also consider two variants of the Ax rule, one more general, one weaker:

$$\frac{}{\varphi, \Gamma \Rightarrow \Delta, \varphi}\ \mathrm{Ax}_{\mathrm{gen}} \qquad\qquad \frac{}{P \Rightarrow P}\ \mathrm{Ax}^{-} \quad (P \text{ atomic})$$

Write $\mathbf{G3}^{-}_{\rightarrow}$ for the system generated by Ax^{-}, $L \rightarrow$, and $R \rightarrow$. The goal in this problem is to show that $\mathbf{G3}^{-}_{\rightarrow}$ does not satisfy the admissibility properties of **G3**.

(a) Show that $\mathrm{Ax}_{\mathrm{gen}}$ is admissible over $\mathbf{G3}_{\rightarrow}$.

(b) Show the inversion principle for $L \rightarrow$, for $\mathbf{G3}^{-}_{\rightarrow}$: if a sequent $\varphi \rightarrow \psi, \Gamma \Rightarrow \Delta$ is derivable, then so are $\Gamma \Rightarrow \Delta, \varphi$ and $\psi, \Gamma \Rightarrow \Delta$.

For the next parts, we use an auxiliary notion: we define the *polarity* (positive or negative) of occurrences of atomic formulas in a formula φ, by recursion on φ:

- when φ is itself an atom P, the only atom occurring in it is P itself, and that occurrence is positive;
- when φ is $\psi \rightarrow \chi$, an atom occurs positively overall if it occurs either positively in χ or negatively in ψ; and negatively, if it occurs negatively in χ or positively in ψ. (More briefly: occurences in χ keep the same polarity; occurrences in ψ get their polarity flipped.)

This is most easily understood by example. In the formula $P \rightarrow Q$, P occurs negatively and Q occurs positively. In $(P \rightarrow Q) \rightarrow (R \rightarrow Q)$, the P is positive, R is negative, and Q occurs both positively and negatively.

Generally, in a sequent $\Gamma \Rightarrow \Delta$, say atoms occur positively if they occur positively in some formula in Δ or negatively in Γ, and negatively if they occur negatively in Δ or positively in Γ. (So like in $\rightarrow$, occurrences in the antecedent get polarity flipped.)

(c) Show that in $\mathbf{G3}^{-}_{\rightarrow}$, if a sequent $\Gamma \rightarrow \Delta$ is derivable, then any atom that occurs positively must also occur negatively. *(Hint: work by induction on derivations.)*

(d) Show that the sequent $P \rightarrow Q \Rightarrow P \rightarrow Q$ is not derivable in $\mathbf{G3}^{-}_{\rightarrow}$, and hence $\mathrm{Ax}_{\mathrm{gen}}$ is not admissible.

(e) Is there any non-atomic formula φ for which $\varphi \Rightarrow \varphi$ is derivable in $\mathbf{G3}^{-}_{\rightarrow}$?

(As usual in such proofs, don't feel obliged to be pedantically thorough: when several cases in a proof are similar, feel free to say so and give just as many cases as you feel illustrate the issues involved.)

——— End of exam — Slut på provet ———