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Exam / Tentamen
Set Theory & Metamathematics
MM7048, 7,5 hp, VT2024
Wed, 2024-05-14

This exam consists of 5 questions, worth a total of 40 points. Note that some subproblems are intentionally quite challenging, and you do not need to complete all questions in order to achieve a good grade. You may submit your answers in either English or Swedish. Write clearly and motivate your answers carefully.

Good luck! — Lycka till!

1. **Ranks of standard algebraic objects.** Give set-theoretic constructions of $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$. Show that they are all in $V_{\omega+\omega}$, and find their precise ranks.

2. **A non-example of an inner model.** Which axioms of ZFC hold in the class On of ordinals?

   (Justify your answers, in particular any absoluteness you rely on. Since On *is not a model of all of ZFC, some "equivalent" formulations of axioms may no longer be equivalent there; in such cases, be clear which formulation of each axiom you consider, and (optionally) discuss alternatives.)*

3. **Initial segments of** L**.** Suppose (κ is a regular cardinal)^L.

   (a) Show that if $X \in L_\kappa$, then $(\|X\| < \kappa)^L$.

   (b) Show that if $Y \in L$, $Y \subseteq L_\kappa$, and $(\|Y\| < \kappa)^L$, then $Y \in L_\kappa$.

   (c) Show that ZF$^-$ (i.e. all of ZF except possibly the power-set axiom) holds in L_κ.

   (To reduce relativisation needed, work in L throughout. Many parts of (c) follow directly from lemmas on L we gave in class; don't worry about re-proving these lemmas, but recall them precisely enough to show they apply here.)

4. **Forcing disjunction.** Recall the case for $\vee$ in the inductive definition of forcing:

$$p \Vdash \varphi(\vec{x}) \vee \psi(\vec{x}) \quad \Leftrightarrow \quad \text{the set of } q \leq p \text{ such that either } q \Vdash \varphi(\vec{x}) \text{ or } q \Vdash \psi(\vec{x}) \text{ is dense below } p.$$

   Give the case for formulas $\varphi \vee \psi$ in the proofs by induction on formulas of each of the following theorems:

   (a) The fundamental theorem of forcing: For all suitable $\mathcal{M}$, $\mathbb{P}$, and $\mathcal{M}$-generic G, and $\mathbb{P}$-names $\vec{x} \in \mathcal{M}^{\mathbb{P}}$,

$$\varphi(\vec{x}_G)^{\mathcal{M}[G]} \quad \Leftrightarrow \quad \exists p \in G \; (p \Vdash \varphi(\vec{x}))^{\mathcal{M}}$$

   (b) The subsequent proposition: For all suitable $\mathcal{M}$, $\mathbb{P}$, and any condition $p \in \mathbb{P}$ and $\mathbb{P}$-names $\vec{x} \in \mathcal{M}^{\mathbb{P}}$,

$$(p \Vdash \varphi(\vec{x}))^{\mathcal{M}} \quad \Leftrightarrow \quad \forall G \subseteq \mathbb{P} \; \mathcal{M}\text{-generic}, p \in G \Rightarrow \varphi(\vec{x}_G)^{\mathcal{M}[G]}$$

5. **Adding a single Cohen real.** We saw in class that forcing with $\mathrm{Fin}(\omega_2 \times \omega, 2)$ breaks CH, blowing up $\mathcal{P}(\omega)$ to ω_2. Generally, forcing with $\mathrm{Fin}(\kappa \times \omega, 2)$ is known as "adding κ Cohen reals" (with the usual set-theoretic perversion of calling subsets of ω "reals"). Here we investigate the consequences of adding a single Cohen real.

   Throughout, assume as usual that $\mathcal{M}$ is a countable transitive model of ZFC.

   (a) Show that for any poset $\mathbb{P} \in \mathcal{M}$ and $\mathcal{M}$-generic $G \subseteq \mathbb{P}$, any subset of ω in $\mathcal{M}[G]$ has some name in $\mathcal{P}(\mathbb{P} \times \{\check{n} \mid n \in \omega\})^M$.

   (b) Show that if $\mathbb{P}$ is moreover *countable*, then $\mathcal{M}[G]$ believes $\|\mathcal{P}(\omega)\| = \|(\mathcal{P}(\omega))^M\|$.

   (c) Conclude that taking $\mathbb{P}$ to be $\mathrm{Fin}(\omega, 2)$, *CH* holds in $\mathcal{M}[G]$ if and only if it holds in $\mathcal{M}$.

   (d) *Optional, not for credit: think about what we can now conclude about adding κ Cohen reals, for larger κ.*

—— End of exam — Slut på provet ——