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Re-exam / Omtentamen
Set Theory & Metamathematics
MM7048, 7,5 hp, VT2024
Tuesday 2024-08-20

This exam consists of 5 questions, worth a total of 40 points. Note that some subproblems are intentionally quite challenging, and you do not need to complete all questions in order to achieve a good grade. You may submit your answers in either English or Swedish. Write clearly and motivate your answers carefully.

Good luck! — Lycka till!

1. **Cardinality calculations for sets of reals**

   (a) Show that if S is any countable set of reals, then $\mathbb{R} \setminus S$ has size $2^{\aleph_0}$.

   (b) Show that the set of irrational numbers has size $2^{\aleph_0}$.

   (c) Show that the set of transcendental real numbers has size $2^{\aleph_0}$. *(A real number is* transcendental *if it is not* algebraic*; that is, if it is not a zero of some polynomial with integer coefficients.)*

2. **Absoluteness.**

   (a) Give the definition of *absoluteness* for formulas: specifically, state what it means for a formula $\varphi(\vec{x})$ to be absolute for a class M.

   For the rest of this question, take "pair" to mean "unordered pair". Recall that "z is the pair $\{x, y\}$", or briefly "$z = \{x, y\}$", is defined as "$\forall\, w\ (w \in z \leftrightarrow (w = x \vee w = y))$".

   (b) Show that the formula "$z = \{x, y\}$" is absolute for transitive classes. *(If desired, you may make use of general facts about absoluteness that we showed in class.)*

   (c) Show that "transitive" is necessary in (b) — that "$z = \{x, y\}$" is not absolute for all classes.

3. **Maximality of** HOD

   (a) Show that for any definable function-class F on the ordinals, the range of F is a subclass of OD. *(Spelled out in detail: for any formula $\varphi(x, y)$, show in ZF(C) that if $\varphi(x, y)$ defines a function on ordinals, in that $\forall x \in \mathrm{On}$, $\exists! y$, $\varphi(x, y)$, then every element of the range of the function defined by $\varphi(x, y)$ is ordinal-definable.)*

   (b) Show that if M is any definable class and $\prec$ is a definable well-ordering of M, then there is a definable surjection from On to M. *(In class we showed this for the specific case of* OD *and its standard well-ordering.)*

   (c) Deduce that OD is the largest definably well-ordered class. That is, if C is any definable class and $\prec$ some definable well-ordering on C, then $C \subseteq \mathrm{OD}$.

   (d) Deduce further that HOD is the largest definably well-ordered transitive class, in a similar sense.

4. **Forcing an implication.** Recall the case for $\to$ in our inductive definition of forcing:

$$p \Vdash \varphi(\vec{x}) \to \psi(\vec{x}) \quad \Leftrightarrow \quad \text{for all } q \leq p, \text{ if } q \Vdash \varphi(\vec{x}) \text{ then } q \Vdash \varphi(\vec{x})$$

   (a) Recall the fundamental theorem of forcing: For all suitable $\mathcal{M}$, $\mathbb{P}$, and $\mathcal{M}$-generic G, and $\mathbb{P}$-names $\vec{x} \in \mathcal{M}^{\mathbb{P}}$,

$$\varphi(\vec{x}_G)^{\mathcal{M}[G]} \quad \Leftrightarrow \quad \exists p \in G \; (p \Vdash \varphi(\vec{x}))^{\mathcal{M}}.$$

   Give the case for $\to$ in the inductive proof of this theorem: that is, assuming this equivalence holds for the formulas $\varphi(\vec{x})$ and $\psi(\vec{x})$, show that it holds for $\varphi(\vec{x}) \to \psi(\vec{x})$.

   (b) More naïvely, one might try to define $p \Vdash \varphi(\vec{x}) \to \psi(\vec{x})$ as simply "if $p \Vdash \varphi(\vec{x})$, then $p \Vdash \psi(\vec{x})$"; write this as $p \Vdash' \varphi(x) \to \psi(\vec{x})$. Compare $p \Vdash' \varphi(x) \to \psi(\vec{x})$ with the correct definition of $p \Vdash \varphi(x) \to \psi(\vec{x})$ above: show that one of them implies the other, but that they are not in general equivalent. *(Hint: for one possible counterexample to show they're not equivalent, you can take the forcing poset* $\mathrm{Fin}(I, \omega)$*, with* $p = \mathbb{1} = \varphi$ *the top element, and consider a suitable negation* $\varphi \implies \bot$*.)*

5. **Forcing with** $\mathrm{Fin}(I \times \omega, 2)$ **vs. with** $\mathrm{Fin}(I, 2^\omega)$**.**

   First consider forcing with the poset $\mathbb{P}_I = \mathrm{Fin}(I \times \omega, 2)$.

   (a) The "canonical new I-indexed family of reals" in a generic extension $M[G]$ is the function $g : I \to \mathcal{P}(\omega)$ corresponding to $\bigcup_{p \in G} p : I \times \omega \to 2$. Explicitly write down a $\mathbb{P}_I$-name for g, i.e. some $x \in M^{\mathbb{P}_I}$ such that $x_G = g$.

   (b) Show that for each $i \in I$, $g(i) \in \mathcal{P}(\omega)$ really is "new", in that $g(i) \notin M$.

   Now consider forcing instead with $\mathbb{P}'_I = \mathrm{Fin}(I, 2^\omega)$.

   (c) Forcing with $\mathbb{P}'_I$ also also adds a "canonical new I-indexed family of reals" $g' : I \to \mathcal{P}(\omega)$; write down a $\mathbb{P}'_I$-name for this g'.

   (d) Show that the overall function g' is new, but its individual values $g'(i)$ are not new.

(One interpretation of these results is that $\mathrm{Fin}(I \times \omega, 2)$ *is really adding a generic function* $I \times \omega \to 2$*, while* $\mathrm{Fin}(I, 2^\omega)$ *is adding a generic function* $I \to 2^\omega$*, and although* $2^{I \times \omega}$ *and* $(2^\omega)^I$ *are bijective as sets, they are different as* spaces*, so their "generic points" may have different properties. There are topological approaches to forcing which make this idea precise.)*

—— End of exam —— Slut på provet ——